1 Introduction

In a short philosophical piece penned in the 1930s, Einstein writes rapturously about the beginnings of Western science:

> We honour ancient Greece as the cradle of western science. She for the first time created the intellectual miracle of a logical system, the assertions of which followed one from another with such rigor that not one of the demonstrated propositions admitted of the slightest doubt. (1934: 164)

Einstein then names the miraculous logical system he has in mind and adds by way of comment:

> This marvellous accomplishment of reason gave to the human spirit the confidence it needed for its future achievements. (1934: 164)

The system Einstein had in mind, you might have guessed, is that of Euclid's geometry in the *Elements* (c. 300 BC). Einstein is a recent figure in a long line of those who have admired the *Elements* as a paragon of mathematical method. Euclid's text took pride of place in at least three brilliant mathematical cultures – ancient Greek, mediaeval Arabic, and early modern European – and was a cornerstone of the school curriculum in the West from the Renaissance until the twentieth century. Hailed as a shining example of the mathematical method, in fact of method *tout court*, the *Elements* spawned hundreds of imitators, not just in geometry but in many other fields too.

So, what is the method of Euclid's *Elements*? Starting from some definitions, postulates, and common notions, Euclid derives the geometry of his day theorem by theorem, in a cumulative manner over the course of thirteen books. Book I's postulates and common notions are as follows:[1]

Postulates

Let the following be postulated:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

[1] For Euclid's text, we have used Heath (1925).

Common Notions

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equals, the wholes are equals.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

Euclid supplements these with twenty-three definitions in Book I (omitted here), including further facts about angles and triangles.

Given the importance of the Euclidean method to the epistemology of mathematics and other fields, it is surprising to find so little attention devoted to it in recent philosophy. Imre Lakatos is one of the very few philosophers of the recent past to have written about what he calls ‘the Euclidean Programme’. We shall use Lakatos’s characterisation as a springboard for ours and adopt his name, abbreviating ‘Euclidean Programme’ as ‘EP’. Epistemologists have, of course, examined foundationalism more generally, but they have neglected its more specific, and historically dominant, instance: the EP as it has been conceived over the centuries. Against this trend, the present essay is devoted to examining the EP.

First of all, we must clarify that the EP is not to be conflated with the axiomatic method in mathematics. The axiomatic method in general is of huge importance, mathematically, historically, and philosophically. And, of course, Euclid’s *Elements* both pioneers the method and is a paradigm of it. But the Euclidean Programme is a particular philosophical take on the axiomatic method and goes beyond mere practice of the method. It will be the focus of our attention here.[2]

As to what the EP actually *is*, we propose a rational reconstruction of its key principles in §2. This reconstruction tries to model what people who have been inspired by the *Elements* have maintained. Like any such reconstruction, ours does not correspond to a historically attested expression; rather, it draws together some key ideas behind various expressions. Although we are more interested in philosophical analysis of the EP than in its long history, a historical overview will nevertheless be useful. We take the apogee of the EP to be in the early modern period, specifically the seventeenth century. We compare our reconstruction of the EP with three historical accounts: Aristotle’s discussion of scientific method in the *Posterior Analytics* (§4), which predates Euclid, and two seventeenth-century versions, in Descartes’ *Discourse on Method* and Pascal’s *On the Geometric Mind* respectively (both in §5). Before that, we say a few words about the *Elements* (§3),

[2] We will not, therefore, be discussing some of the most important figures in axiomatic mathematics (e.g. David Hilbert), or some of its most important features, such as the organisation of a mathematical subfield, in significant detail.

the main point being to caution the reader against confusing the EP with how Euclid actually proceeds in the *Elements*. We conclude the more historical discussion with a schematic twentieth-century account of Descriptive Axiomatisation (§6). History is then followed by philosophical assessment. §§7–8 critically assess the EP, and §9 sketches what should replace it. §10 very briefly concludes.

All in all, the present essay offers a combined historical and critical analysis of the Euclidean Programme.[3] We try to impose some structure on a historical jumble of ideas (§§3–6), but we also argue for a position about the EP's current status (§§7–9).

2 The Euclidean Programme

The term 'Euclidean Programme' is borrowed from Lakatos, whose paper prompted our interest in delineating it. Lakatos contrasts the Euclidean Programme with an Empiricist one:

> The Euclidean programme proposes to build up Euclidean theories with foundations in meaning and truth-value at the top, lit by the *natural light of Reason*, specifically by arithmetical, geometrical, metaphysical, moral, etc. intuition. The Empiricist programme proposes to build up Empiricist theories with foundations in meaning and truth-value at the bottom, lit by the *natural light of Experience*. Both programmes however rely on Reason (specifically on logical intuition) for the safe transmission of meaning and truth-value. (Lakatos 1962: 5)

We return to the Empiricist Programme in §9 and until then concentrate on the Euclidean one. Lakatos describes the latter in more detail in the following passage:

> I call a deductive system a 'Euclidean theory' if the propositions at the top (*axioms*) consist of perfectly well-known terms (*primitive terms*), and if there are *infallible truth-value-injections* at this top of the truth-value *True*, which flows downwards through the deductive channels of truth-transmission (*proofs*) and inundates the whole system. (If the truth-value at the top was *False*, there would of course be no current of truth-value in the system.) Since the Euclidean programme implies that all knowledge can be deduced from a finite set of trivially true propositions consisting only of terms with a trivial meaning-load, I shall call it also the *Programme of Trivialization of Knowledge*. Since a Euclidean theory contains only indubitably true propositions, it operates neither with conjectures nor with refutations. In a fully-fledged Euclidean theory meaning, like truth, is injected at the top and it flows down safely through meaning-preserving channels of nominal definitions from the primitive terms to the (abbreviatory and therefore theoretically superfluous) defined terms. A Euclidean theory is *eo ipso* consistent, for all the propositions occurring in it are true, and a set of true propositions is certainly consistent. (Lakatos 1962: 4–5)

[3] Other authors have used different names for what, following Lakatos, we call the Euclidean Programme. For example, Shapiro (2009: 181) calls it *Euclidean Foundationalism*.

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In this passage, Lakatos speaks of truth and meaning injection, but this is somewhat misleading. The EP represents an epistemological conception, and the hierarchical path from axioms to theorems is a path followed by a subject.[4] The flow metaphor is better construed as the transmission of an epistemic good of some sort, such as justification. The picture is then a foundationalist one in which one gains justification for the axioms first and thence for theorems by inferring these from the axioms.

Lakatos also calls the axioms 'trivially true' and says they bear a 'trivial meaning-load'. We don't know what exactly Lakatos meant by the word 'trivial'. One way to understand it is as the broadly empiricist idea, favoured by Hume and the logical empiricists: mathematical statements are true in virtue of meaning and therefore empty of content.[5] If so, we part ways with Lakatos: it is entirely compatible with the EP that axioms are substantive. For example, recognition of the axioms' truth could be the product of mathematical intuition, a faculty distinct from any that informs us of the trivial truth of statements such as 'bachelors are unmarried'. We take the idea at the heart of the EP to be that axioms are self-evident, and we remain neutral on their 'triviality' (whatever exactly this means).

Lakatos then goes on to make a very acute observation – the key, we believe, to understanding the Euclidean Programme:

> We can get a long way merely by discussing *how* anything flows in a deductive system without discussing the problem of *what in fact flows* there, infallible truth or only, say, Russellian 'psychologically incorrigible' truth, Braithwaitian 'logically incorrigible' truth, Wittgensteinian 'linguistically incorrigible' truth or Popperian corrigible falsity and 'verisimilitude', Carnapian probability. (1962: 6)

Earlier, we spoke of an epistemic good flowing from axioms to theorems. This is the right way to characterise the EP if we are to maintain generality and avoid, or at least minimise, anachronism. The insight we extract from Lakatos is that we can achieve this by considering *how* the epistemic good flows rather than *what* it is. Succinctly, the EP is all about *Euclidean hydraulics*. An analogy: think of the Phillips machine, a post-war hydraulic model of the economy. Its inventor, Bill Philips, used it to demonstrate how money moves through an economy by letting coloured water flow through clear pipes.[6] In our epistemic analogue, the coloured water corresponds to the epistemic good. It is injected at the top, where the axioms lie, and thence flows down to the theorems.

4 It is unclear what it might literally mean for truth and meaning to flow down some channel.
5 If axioms were so trivial as to be logical then they would be unnecessary, as they would be delivered by the logic. But we take it Lakatos has a broader sense of triviality in mind.
6 Readers should google 'MONIAC' for a demonstration of the machine at work.

Different versions of the EP will differ on what exactly the good is. We note, however, that not every epistemic good possessed by the axioms will flow down the relevant channels; in particular the axioms' *self-evidence* (about which more shortly) may not be transferred to the theorems by inference.

Another important point is that our understanding of a theory is broader than that of the contemporary logician, who understands it, roughly, as a deductively closed set of formal sentences in a formal logic. As we will see, the language of a theory putatively instantiating the EP does not have to be formal; it could be Greek, English, or any other language. It should be no part of the EP that an axiomatisation be formal, as that would be unfaithful to its history. Indeed, as Jonathan Barnes points out, the idea of a formal language was alien to ancient deductive thought.[7]

Nor do inferences in this context have to be purely logical.[8] Kantians might, for example, maintain that mathematical reasoning employs ineliminably mathematical modes of inference (say, spatial intuition in geometry); if so, the conclusion is in a strong, but not strictly logical, sense implied by the premises. So as not to restrict the EP's range of application too narrowly, we allow inferences that track these implications as part of the Euclidean picture. Moreover, which inferences one considers logical will be sensitive to the background logic, and the EP does not prescribe a particular background logic to be used. Indeed, advocacy of the EP is perfectly consistent with some version of an anti-logical view, as espoused by Descartes, for example (see §5.1). In short: a theory for us is simply a collection of sentences about a subject matter, closed under a relation that need not be formal or even logical.

To further clarify this point, consider the Kneales' account of the geometric method in their classic text *The Development of Logic*. The Kneales single out three ingredients in the 'customary presentation of geometry as a deductive science' (1962: 3). First, 'certain propositions of the science must be taken as true without demonstration'; second, 'all the other propositions of the science must be derived from these' (1962: 3). The last ingredient is at once the most distinctive and the most controversial of the three:

> … the derivation must be made without any reliance on geometrical assertions other than those taken as primitive, i.e. it must be *formal* or independent of the special subject matter discussed in geometry … [thus] elaboration of a deductive system involves consideration of the relation of logical consequence or entailment. (1962: 3–4)

7 '[N]either they [the Stoics] nor any other ancient logician ever considered inventing an artificial language for the use of logic' (Barnes 2005: 512).

8 Like others, we distinguish inference (a movement in thought) from implication or consequence (a relation among propositions). In this Element, we are almost exclusively interested in the former.

Kneale and Kneale do not clarify whether the 'or' in '*formal* or independent of the special subject matter' is supposed to present two alternatives (the second condition being different) or just one (the second spelling out the first). Whatever they intended, the idea that the deductive logic in any axiomatic presentation of geometry must be formal should be resisted. Even if, as we believe, logic is formal, it should be no requirement of a Euclidean account of geometry that its logic be formal. (Euclid's logic itself was certainly not, though in §3 we shall draw a contrast between the flesh-and-blood Euclid and the ideal he imperfectly manifested.)

What *is* clear is that Kneale and Kneale insist on derivations being strictly logical. But it is equally clear that they mean to characterise any 'customary' axiomatic theory, including Euclid's. To stipulate that such an axiomatisation's rules *must* be strictly logical seems too stringent a requirement; it risks, for example, making the *Elements* not a 'customary' axiomatisation, if Euclid's system is not strictly logical because it appeals to geometric insight in various places (see §3). More generally, there is no strong historical precedent, prior to the late nineteenth century at any rate,[9] for thinking that the rules in a Euclidean axiomatisation may not be topic-specific. It is better, then, to characterise rules more neutrally and not decree that they be formal, or strictly logical.

Returning to Lakatos, we note that, for him, primitive terms of a theory must be *perfectly* well-known (again, in virtue of their meaning being somehow trivial). As we see it, however, the Euclidean Programme is primarily an epistemology of mathematical propositions, not terms. Given the axioms' pride of place in the EP, our understanding of the primitive terms must be sufficiently clear to enable the mathematician to understand, and hence see the truth of, the axioms. But there is little justification for Lakatos's assertion that the primitive terms of a Euclidean theory must be perfectly understood, for this is not required for the axioms to be self-evident. It can be completely evident, for example, that anybody taller than a tall person is tall, even to someone with a less-than-perfect understanding of the predicate 'is tall'. Or, for a mathematical example, it would have been completely evident to an eighteenth-century mathematician that the identity mapping on the reals was a function, even in the absence of a clear understanding of what real-valued functions, or even the reals, are. So, we require only that the axioms be graspable, in the sense of being possible to understand, and self-evident to a mathematician who has grasped the meanings of the primitive terms to an extent which allows them to understand the axioms, whether or not their grasp of the primitive terms is perfect.

9 The slightly oblique reference here is to Frege, who believed that the rules were (as we would now put it) topic-neutral.

To continue the hydraulic metaphor, we can tolerate some impurity in the water, so long as it does not affect the flow.

With all that in mind, let's try to express the general picture slightly more precisely. At the picture's core is a three-place epistemic relation relating a subject S to a proposition p to a certain degree d: we might formalise this as $E(S, p, d)$. ('E' is nicely suggestive of both 'Euclidean' and 'epistemic'.) There is a suppressed time index here, which we usually ignore, since it won't affect the discussion much. We take p to be a proposition, but with some minor adjustments it could equally well be taken to be a belief, or even a fact. Relation E is a placeholder for a more specific epistemic relation, which different proponents of the EP will want to construe in different ways, say as some species of justification or warrant. Talk of the subject's having the relevant 'epistemic good' is then simply another way of saying that the subject stands in this relation E (to p and to degree d). For some p, the subject S may stand in relation E to p to the maximal degree – call this *max*. This, according to the EP, is the case for the axioms, so long as S clearly grasps them. (Different versions of the EP will have a different story to tell about what clearly grasping the axioms amounts to.) As an illustration, if we equate E with justified belief this becomes: S is maximally justified in believing any axiom a. (A notion which in turn can be made more precise in different ways, depending on the precise type of justification in question.) In a limiting case, which our characterisation allows for but does not focus on, justification could be all or nothing, so that there are just two degrees. Moreover, axioms must, of course, be true, as must be the sentences inferred from them.

Next, the EP contains a principle governing E-flow, or transmission of the relevant epistemic good E. In a strong version, the degree d is preserved in an inference from the conjunction of an inference's premises to its conclusion; in a weaker version, it is more or less preserved. A special case of the strong version is when the subject S is in the highest epistemic state with respect to the (finitely many) axioms' conjunction A and thus, according to the principle, potentially so with respect to the theorems. In that case, if $E(S, A, max)$ and p follows from A, then S can reason her way to p from A using the appropriate rules; and if she does so then $E(S, p, max)$ will hold. The weaker version of the flow principle is that in such a case if $E(S, A, d)$ then $E(S, p, d^*)$ obtains for some d^* not much lower than d. The weaker version of the flow principle is vague, and deliberately so. Being too precise about d-transmission here would be anachronistic and risk obscuring important common features between different historical expressions of the EP. Instantiated by justification, the weaker version of the principle says that our justification for theorems derived in this way is high; but it allows that this justification may not be maximally high, allowing for some erosion of justification in the course of inferring theorems (about which more in §7.3).

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We assume throughout that the subject *S* has no other epistemic access to the conclusion than that provided by inferring it from the axioms. So we ignore for example the following sort of case: *S* infers *p* from some premises, *S* knows that *p* is Emmy Noether's favourite theorem and also knows that Noether was a highly reliable mathematician. *S* might then legitimately be more confident in *p* than in the conjunction of the inference's premises. More generally, we ignore testimonial and other sources of evidence, the better to focus on EP's epistemology of *proof*. Another complication we largely ignore (although see §7.3) are cases in which *S* reasons to the same *p* in different ways – via different proofs – which might result in *S* standing in relation *E* to *p* to a higher degree than if *S* just reasoned to *p* in only one such way.

We are now ready to present our rational reconstruction of the EP, which is made up of three core principles and four further ones. This simple device will permit a thoroughgoing comparison of diverse historical figures in the Euclidean tradition and facilitate a comparison of their actual methodology to this reconstructed ideal. Of course, any relation between the two is bound to be loose and inexact; a perfect fit is not to be expected. The historian of philosophy must be careful to avoid attributing claims to past philosophers in terms they would not acquiesce to. Our aim is to relate the EP, stated *in vacuo*, to real historical conceptions. Although the point of the exercise is to show that the EP does relate interestingly to various historical expressions of 'Euclideanism', we must be careful not to confuse an abstract prototype with historical expressions that suggest or approximate it in some interesting fashion. Having said that, something would be amiss with our rational reconstruction if it did *not* display important similarities with these historical expressions, especially the seventeenth-century ones.

Delaying the historical comparisons for now, we summarise the three core principles of the EP as follows:

EP-Truth	All axioms and theorems are true.
EP-Self-Evidence	All axioms are self-evident. That is to say, they are all graspable and if a subject clearly grasps an axiom then she bears relation *E* to it to the maximal degree.
EP-Flow	If a conclusion follows from some premises, and the subject clearly grasps this, and bears relation *E* to these premises to a high degree, she thereby bears relation *E* to the conclusion to the same, or a similarly high, degree.

To reiterate a key point, it is crucial that the relation *E* not be further specified, to make the EP an umbrella conception large enough to cover many and varied historical instances. Choosing a specific relation for *E* would rule out some