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# Intersection Theory of Matroids: Variations on a Theme

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**Abstract**

Chow rings of toric varieties, which originate in intersection theory, feature a rich combinatorial structure of independent interest. We survey four different ways of computing in these rings, due to Billera, Brion, Fulton–Sturmfels, and Allermann–Rau. We illustrate the beauty and power of these methods by giving four proofs of Huh and Huh–Katz's formula $\mu^k(\mathsf{M}) = \deg_{\mathsf{M}}(\alpha^{r-k}\beta^k)$ for the coefficients of the reduced characteristic polynomial of a matroid $\mathsf{M}$ as the mixed intersection numbers of the hyperplane and reciprocal hyperplane classes $\alpha$ and $\beta$ in the Chow ring of $\mathsf{M}$. Each of these proofs sheds light on a different aspect of matroid combinatorics, and provides a framework for further developments in the intersection theory of matroids.

Our presentation is combinatorial, and does not assume previous knowledge of toric varieties, Chow rings, or intersection theory. This survey was prepared for the Clay Lecture to be delivered at the 2024 British Combinatorics Conference.

## 1 Introduction

Our starting point is the *chromatic polynomial* $\chi_G(t)$ of a graph $G = (V, E)$. For a positive integer $q$,

$$\chi_G(q) := \text{ number of proper vertex-colorings of G with } q \text{ colors},$$

where a coloring is *proper* if no two neighboring vertices have the same color. For example, the chromatic polynomial of the graph below is $\chi_G(q) = q(q-1)^2(q-2)$.

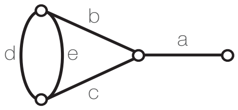



Figure 1: A graph $G$ with $\chi_G(q) = q^4 - 4q^3 + 5q^2 - 2q$. and $\mu^0 = 1, \mu^1 = 3, \mu^2 = 2$.

More generally, the *characteristic polynomial* $\chi_{\mathsf{M}}(t)$ of a matroid $\mathsf{M} = (E, r)$ is

$$\chi_{\mathsf{M}}(q) := \sum_{A \subseteq E} (-1)^{|A|} q^{r-r(A)}. \tag{1.1}$$

It is one of the most important invariants of a matroid; it is introduced in detail in Section 3 and [4, Sections 6, 7]. The characteristic polynomial generalizes the chromatic polynomial in the sense that if $\mathsf{M}(G)$ is the cycle matroid of a graph $G$ that has $c$ connected components, then $\chi_G(q) = q^c \chi_{\mathsf{M}(G)}(q)$. This polynomial is a multiple of $q-1$, and we define the *reduced characteristic polynomial* of $\mathsf{M}$ to be

$$\overline{\chi}_{\mathsf{M}}(q) := \frac{\chi_{\mathsf{M}}(q)}{q-1} = \mu^0 q^r - \mu^1 q^{r-1} + \cdots + (-1)^r \mu^r q^0$$

where $r+1$ is the rank of $\mathsf{M}$. In the example above, $\overline{\chi}_{\mathsf{M}}(q) = q^2 - 3q + 2$.

It is not too difficult to prove recursively that the numbers $\mu^0, \mu^1, \ldots, \mu^r$ are non-negative. A combinatorialist then asks: Do they count something? An algebraic combinatorialist then asks: Do they have an algebraic, geometric, or topological interpretation? Such questions often give rise to a deeper understanding of the objects under study. In this case, and in numerous others, they lead to proofs of long-standing conjectures for which no purely combinatorial proof is known.

## 1.1 Theme

Our recurring theme will be Huh [31] and Huh-Katz [34]'s remarkable interpretation of $\mu^0, \ldots, \mu^r$. Beautiful in its own right, their Theorem 1.1 also lies at the heart of the celebrated proof of the conjecture that this sequence is log-concave [1].

Let $\mathsf{M}$ be a matroid of rank $r+1$ on a set $E$ with $n+1$ elements. The *Chow ring* $A(\mathsf{M})$ is the $\mathbb{Z}$-algebra generated by variables $x_F$ for each non-empty proper flat, with relations

$$x_F x_G = 0 \qquad \text{for any flats } F, G \text{ such that } F \subsetneq G \text{ and } F \supsetneq G,$$
$$\sum_{F \ni i} x_F = \sum_{F \ni j} x_F \qquad \text{for any elements } i, j \in E.$$

One can show that the Chow ring is graded $A(\mathsf{M}) = A^0(\mathsf{M}) \oplus \cdots \oplus A^r(\mathsf{M})$, and that there is a canonical isomorphism $\deg_{\mathsf{M}} : A^r(\mathsf{M}) \xrightarrow{\sim} \mathbb{R}$ called the *degree map* [1].

Consider the following two elements of $A^1(\mathsf{M})$, which we call the *hyperplane* and *reciprocal hyperplane* classes:

$$\alpha = \alpha_i = \sum_{i \in F} x_F, \qquad \beta = \beta_i = \sum_{i \notin F} x_F.$$

One readily verifies that they do not depend on $i$.

**Theorem 1.1** *Let $\mathsf{M}$ be a matroid of rank $r+1$. Let $\alpha, \beta$ be the hyperplane and reciprocal hyperplane classes in the Chow ring $A(\mathsf{M})$. Then*

$$\deg_{\mathsf{M}}(\alpha^{r-k}\beta^k) = \mu^k(\mathsf{M}) \qquad \text{for } 0 \leq k \leq r.$$

The Chow ring $A(\mathsf{M})$ has remarkable Hodge-theoretic properties [1] surveyed in [33, 5, 13, 25]. In particular, $A(\mathsf{M})$ satisfies the *Hodge–Riemann relations*, which give

$$\deg_{\mathsf{M}}(\ell_1\ell_2\ell_3\cdots\ell_d)^2 \geq \deg_{\mathsf{M}}(\ell_1\ell_1\ell_3\cdots\ell_d)\deg_{\mathsf{M}}(\ell_2\ell_2\ell_3\cdots\ell_d),$$

for any $\ell_1, \ell_2, \ldots, \ell_d$ in a certain cone $\mathcal{K}(\mathsf{M}) \subseteq A^1(\mathsf{M})$ whose closure contains $\alpha$ and $\beta$. In light of Theorem 1.1, this proves the following inequalities conjectured by Rota [46], Heron [30], and Welsh [50] in the 1970s:

$$(\mu^k)^2 \geq \mu^{k+1}\mu^{k-1} \qquad \text{for } 1 \leq k \leq r-1.$$

This survey focuses on the combinatorial aspects of this program:

**Question 1.2** *How does one discover and prove combinatorially interesting formulas in Chow rings like Theorem 1.1?*

This question fits within the framework of intersection theory of toric varieties, in ways that can be understood combinatorially. The Chow ring $A(X_\Sigma)$ of a toric variety $X_\Sigma$ corresponding to a rational polyhedral fan $\Sigma$ is a beautifully rich object that can be understood from several different points of view. We will present four, due to Billera, Brion, Fulton–Sturmfels, and Allermann–Rau [16, 18, 28, 3]. Each of these points of view gives us a different ways to compute in a Chow ring, and teaches us different things about the objects at hand. This machinery is relevant to Theorem 1.1 because the Chow ring of a matroid $\mathsf{M}$ equals the Chow ring of the toric variety $X_{\Sigma_\mathsf{M}}$ and is closely related to the permutahedral toric variety $X_{\Sigma_E}$, where $\Sigma_E$ and $\Sigma_\mathsf{M}$ are the *matroid fan* of $\mathsf{M}$ and the *braid fan* of $E$, discussed in detail in Sections 2.0 and 3.0.

Our presentation will be combinatorial, and will not assume previous knowledge of toric varieties, Chow rings, or intersection theory. A familiarity with the basics of enumerative matroid theory will be helpful; see for example [4, 17, 44].

This survey is organized as follows. In Section 2 we discuss the general intersection theory of simplicial rational fans $\Sigma$ and toric varieties $X_\Sigma$, giving four different combinatorial points of view on the Chow ring $A(\Sigma) = A(X_\Sigma)$. We pay special attention to the Chow ring of the braid fan $\Sigma_E$ for a finite set $E$. In Section 3 we discuss some basic aspects of the intersection theory of matroids. The general theory gives us four different ways to think about the Chow ring of a matroid $\mathsf{M}$. We illustrate each one of these approaches by using it to give a different proof of Theorem 1.1.

## 2 Intersection Theory of Toric Varieties: A Case Study

Intersection theory studies how subvarieties of an algebraic variety $X$ intersect. For example, Bezout's theorem tells us that two generic plane curves of degrees $m$ and $n$ intersect at $mn$ points. We want a robust theory that will keep track of multiplicities correctly, and where the answer to such intersection questions does not change under rational equivalence. The Chow ring $A(X)$ provides an algebraic framework to carry out such computations. Because this ring encodes the answers to very subtle questions, it is generally an unwieldy object.

The situation is much better behaved when $X = X_\Sigma$ is the toric variety associated to a simplicial rational fan $\Sigma$. In this case, the Chow ring $A(X_\Sigma)$ can be described entirely in terms of the fan $\Sigma$ in several ways. This leads to algebraic, geometric, and combinatorial methods for computing in $A(X_\Sigma)$, and to combinatorial results of independent interest. Those methods and results are the subjects of this survey.

Let $\mathrm{N}_\mathbb{Z} \cong \mathbb{Z}^n$ be a lattice and $\mathrm{N} = \mathbb{R} \otimes \mathrm{N}_\mathbb{Z} \cong \mathbb{R}^n$ the corresponding real vector space. A *rational cone* $\{\lambda_1 \mathbf{v}_1 + \cdots + \lambda_k \mathbf{v}_k \,:\, \lambda_1, \ldots, \lambda_k \geq 0\}$ is a cone in N generated by finitely many lattice vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathrm{N}_\mathbb{Z}$; it is *strongly convex* if it contains no lines. A *rational fan* $\Sigma$ in N is a set of strongly convex rational cones that are glued along common faces; that is, any face of a cone in $\Sigma$ is a cone in $\Sigma$, and the intersection of any two cones in $\Sigma$ is a cone in $\Sigma$. We say a fan $\Sigma$ is *simplicial* if every $d$-dimensional cone is generated by $d$ vectors, *unimodular* if those $d$ vectors always form a basis for $\mathrm{N}_\mathbb{Z}$, and *complete* if the union of the cones in $\Sigma$ is all of N. We say $\Sigma$ is *pure* if all maximal cones have the same dimension, and write $\Sigma(d)$ for the set of $d$-dimensional cones. A rational fan $\Sigma$ in N determines a toric variety $X = X_\Sigma$;

for details see [20, 27].

The goal of this section is to explain the following theorem. After explaining each of its parts, we use it to compute explicitly the Chow ring of the two-dimensional braid fan.

**Theorem 2.1** *Let $\Sigma$ be a complete simplicial rational fan in $\mathrm{N} = \mathbb{R} \otimes \mathrm{N}_{\mathbb{Z}}$. The following rings are isomorphic:*

1. *The quotient $A(\Sigma) = S(\Sigma)/(I(\Sigma) + J(\Sigma))$ where*

$$\begin{aligned} S(\Sigma) &= \mathbb{Q}[x_\rho : \rho \text{ is a ray of } \Sigma]/(I(\Sigma)+J(\Sigma)), \\ I(\Sigma) &= \langle x_{\rho_1} \cdots x_{\rho_k} : \rho_1, \ldots, \rho_k \text{ do not generate a cone of } \Sigma \rangle, \\ J(\Sigma) &= \langle \sum_{\rho \text{ ray of } \Sigma} \ell(\mathbf{e}_\rho) x_\rho : \ell \text{ is a linear function on } \mathrm{N} \rangle. \end{aligned}$$

2. *The ring $\mathrm{PP}(\Sigma)/\langle \mathrm{N}^\vee \rangle$ of piecewise polynomial functions on $\Sigma$ modulo the ideal generated by the space $N^\vee$ of (global) linear functions on $N$.*
3. *The ring $\mathrm{MW}(\Sigma)$ of Minkowski weights on $\Sigma$ under stable intersection.*
4. *The ring $\mathrm{MW}(\Sigma)$ of Minkowski weights on $\Sigma$ under tropical intersection.*
5. *The cohomology ring of the toric variety $X(\Sigma)$.*
6. *The Chow ring of the toric variety $X(\Sigma)$.*

*When $\Sigma$ is unimodular, these isomorphisms also hold over $\mathbb{Z}$.*

## 2.0 The Braid Fan

For a finite set $E$, we let $\{\mathbf{e}_i : i \in E\}$ be the standard basis of $\mathbb{R}^E$, and we write

$$\mathbf{e}_S := \sum_{s \in S} \mathbf{e}_s \qquad \text{for } S \subseteq E.$$

The fans considered in this paper will live in $\mathrm{N}_E := \mathbb{R}^E / \mathbb{R}\,\mathbf{e}_E$. The image of $\mathbf{e}_S \in \mathbb{R}^E$ in this quotient will also be denoted $\mathbf{e}_S \in \mathrm{N}_E$. We will often consider $E = [0, n] := \{0, 1, \ldots, n\}$.

**Definition 2.2** Let $E$ be a finite set. The *braid fan* $\Sigma_E$ in $\mathrm{N}_E := \mathbb{R}^E / \mathbb{R}\,\mathbf{e}_E$ has

- rays: $\mathbf{e}_S$ for the nonempty proper subsets $\emptyset \subsetneq S \subsetneq E$
- cones: $\sigma_{\mathcal{S}} = \mathrm{cone}(\mathbf{e}_{S_1}, \ldots, \mathbf{e}_{S_k})$ for the flags $\mathcal{S} = (\varnothing \subsetneq S_1 \subsetneq \cdots \subsetneq S_k \subsetneq E)$

The braid fan is the decomposition of $\mathrm{N}_E$ determined by the *braid arrangement* in $\mathrm{N}_E$, which consists of the hyperplanes $t_i = t_j$ for $i, j \in E$. If $|E| = n+1$, the braid fan $\Sigma_E$ is $n$-dimensional, and has a facet $\sigma_{\mathcal{S}} = \sigma_\pi = \{\mathbf{t} \in \mathrm{N}_E : t_{s_0} \geq t_{s_1} \geq \cdots \geq t_{s_n}\}$ for each complete flag $\mathcal{S} = (\varnothing \subsetneq \{s_0\} \subsetneq \cdots \subsetneq \{s_0, s_1, \ldots, s_{n-1}\} \subsetneq E)$, or equivalently, each bijection $\pi : [0, n] \to E$ given by $\pi(i) = s_i$. Slightly abusing terminology, we will call $\pi$ a *permutation* of $E$ and write $\pi = s_0 \dots, s_n$. It follows that the braid fan is complete, simplicial, and unimodular.

Figure 2 shows the braid fan $\Sigma_E$ for $E = [0, 2] = \{0, 1, 2\}$. It is the complete fan in $\mathrm{N}_E$ cut out by the braid arrangement consisting of the lines $t_0 = t_1, t_1 = t_2$, and $t_2 = t_0$ in $\mathrm{N}_E$.

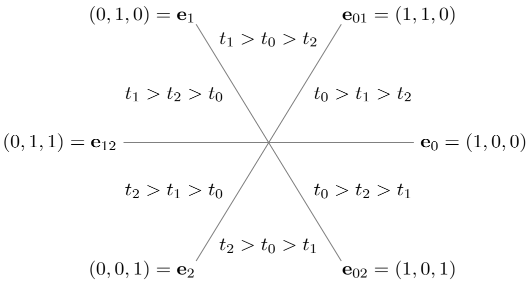



Figure 2: The braid fan $\Sigma_{[0,2]}$.

We will return to this picture many times in what follows; the reader may wish to keep it within reach. We will call its toric variety and Chow ring the *permutahedral variety* and the *permutahedral Chow ring*.

### 2.1 The Chow Ring as a Quotient of a Polynomial Ring

For the remainder of Section 2, $\Sigma$ will be a simplicial rational fan in $\mathrm{N} = \mathbb{R} \otimes \mathrm{N}_{\mathbb{Z}}$.

**The Chow Ring** The *Chow ring* of $\Sigma$ is the graded algebra

$$A(\Sigma) \coloneqq S(\Sigma)/(I(\Sigma) + J(\Sigma)),$$

where

$$\begin{aligned} S(\Sigma) &= \mathbb{Z}[x_\rho \,:\, \rho \text{ is a ray of } \Sigma]/(I(\Sigma) + J(\Sigma)), \\ I(\Sigma) &= \langle x_{\rho_1} \cdots x_{\rho_k} \,:\, \rho_1, \dots, \rho_k \text{ do not generate a cone of } \Sigma \rangle, \\ J(\Sigma) &= \langle \sum_{\rho \text{ ray of } \Sigma} \ell(\mathbf{e}_\rho) x_\rho \,:\, \ell \text{ is a linear function on } \mathrm{N} \rangle. \end{aligned}$$

The ideal $I(\Sigma)$ is called the *Stanley-Reisner ideal* of $\Sigma$ and $S(\Sigma)/I(\Sigma)$ is called its *Stanley-Reisner ring*. In $J(\Sigma)$, it is sufficient to let $\ell$ range over a basis of the space $\mathrm{N}^\vee$ of linear functions on $\mathrm{N}$.

**Example 2.3** (The Chow ring $A(\Sigma_{[0,2]})$.) Let us compute the Chow ring of the braid fan $\Sigma_E$ for $E = [0, 2]$. We have

$$\begin{aligned} S(\Sigma_E) &= \mathbb{R}[x_0, x_1, x_2, x_{01}, x_{02}, x_{12}] \\ I(\Sigma_E) &= \langle x_i x_j : i \neq j \rangle + \langle x_i x_{jk} : i, j, k \text{ distinct} \rangle + \langle x_{ij} x_{jk} : i, j, k \text{ distinct} \rangle \\ J(\Sigma_E) &= \langle (x_0 + x_{02}) - (x_1 + x_{12}), (x_0 + x_{01}) - (x_2 + x_{12}) \rangle \end{aligned}$$

where we use $t_0 - t_1$ and $t_0 - t_2$ as a basis for $\mathrm{N}^\vee$ in the description of $J(\Sigma_E)$.

We claim that $A = A(\Sigma_E)$ has degree 2 and

$$A^0 = \mathbb{Z}\{1\} \cong \mathbb{Z}^1, \qquad A^1 = \mathbb{Z}\{x_0, x_1, x_2, x_{12}\} \cong \mathbb{Z}^4, \qquad A^2 = \mathbb{Z}\{x_0x_{01}\} \cong \mathbb{Z}\,.$$

The description of $A^0$ is clear. The description of $A^1$ follows from the two linear relations in $J(\Sigma_E)$ that express $x_{01}$ and $x_{02}$ in terms of the four chosen generators. To compute $A^2$, notice that

$$x_0^2 = x_0(x_2 + x_{12} - x_{01}) = -x_0x_{01}, \qquad x_{01}^2 = x_{01}(x_2 + x_{12} - x_0) = -x_0x_{01},$$

and similarly for the squares of the other terms $x_ix_{ij}$. This implies that

$$-x_0^2 = -x_1^2 = -x_2^2 = -x_{01}^2 = -x_{02}^2 = -x_{12}^2 = x_ix_{ij} \text{ for all } i \neq j. \tag{2.1}$$

Thus $A^2$ is indeed generated by $x_0x_{01}$, and we have an isomorphism

$$\deg : A^2 \simeq \mathbb{Z}, \qquad \deg(x_ix_{ij}) = 1 \text{ for all facets } \sigma_{i\subset ij} \text{ of } \Sigma_{[0,2]}.$$

Any monomial of degree 3 can be reduced via (2.1) to a square free monomial of degree 3, which is in $I(\Sigma_E)$ and hence vanishes in $A(\Sigma_E)$.

**Computing Degrees** When $\Sigma$ is complete, the Chow ring $A(\Sigma)$ is graded of degree $n$, and there is a canonical *degree map* $\deg : A^n(\Sigma) \simeq \mathbb{Z}$. If $\Sigma$ is unimodular, this map is characterized by the property that the degree of any facet monomial is 1: $\deg(x_\sigma) = 1$ for any facet $\sigma$, where $x_\sigma = \prod_{\rho \text{ ray}} x_\rho$. Any $f \in A^n(\Sigma)$ can be expressed as a linear combination of facet monomials [1, Prop. 5.5], and this expression gives the degree of $f$.

**The Hyperplane and Reciprocal Hyperplane Classes** We will pay special attention to two special elements $\alpha, \beta$ in the degree one piece $A^1(\Sigma_E)$ of the permutahedral Chow ring:

$$\alpha \coloneqq \alpha_i = \sum_{i\in S} x_S, \qquad \beta \coloneqq \beta_i = \sum_{i\notin S} x_S, \qquad \text{for } i \in E.$$

We invite the reader to check that these do not depend on the choice of $i \in E$.

**Example 2.4** (The degree of $\alpha\beta$ in $A(\Sigma_{[0,2]})$.) For $E = [0,2]$ we have

$$\begin{array}{rclcl} \alpha & = & \alpha_0 & = & x_0 + x_{01} + x_{02} \\ & = & \alpha_1 & = & x_1 + x_{01} + x_{12} \\ & = & \alpha_2 & = & x_2 + x_{02} + x_{12} \end{array} \qquad \begin{array}{rclcl} \beta & = & \beta_0 & = & x_1 + x_2 + x_{12} \\ & = & \beta_1 & = & x_0 + x_2 + x_{02} \\ & = & \beta_2 & = & x_0 + x_1 + x_{01}. \end{array}$$

Let us compute the intersection degree of $\alpha$ and $\beta$. Using the relations in the Chow ring, we can write

$$\alpha\beta = \alpha_0\beta_0 = (x_0 + x_{01} + x_{02})(x_1 + x_2 + x_{12}) = x_1x_{01} + x_2x_{02}.$$

This implies that

$$\deg(\alpha\beta) = 2.$$

Note that a different choice of representatives, such as $\alpha_0\beta_1$, leads to a more complicated computation.

## 2.2 The Chow Ring in Terms of Piecewise Polynomials

**The Chow Ring** A *piecewise polynomial* on $\Sigma$ is a continuous function on N whose restriction to each cone in $\Sigma$ agrees with a polynomial function. Let $\mathrm{PP}(\Sigma)$ be the ring of piecewise polynomials on $\Sigma$, with pointwise addition and multiplication. Let $\langle \mathrm{N}^\vee \rangle$ be the ideal of $\mathrm{PP}(\Sigma)$ generated by the set $\mathrm{N}^\vee$ of (global) linear functions on N. Thanks to work of Billera [16], the Chow ring of $\Sigma$ can be described as:

$$A(\Sigma) \cong \mathrm{PP}(\Sigma)/\langle \mathrm{N}^\vee \rangle.$$

**The Dictionary** Billera [16] constructed an isomorphism from the Stanley-Reisner ring $S(\Sigma)/I(\Sigma)$ to the algebra $\mathrm{PP}(\Sigma)$ of continuous piecewise polynomial functions on $\Sigma$, by identifying the variable $x_\rho$ with the piecewise linear *Courant function* on $\Sigma$ determined by the condition

$$x_\rho(\mathbf{e}_{\rho'}) = \begin{cases} 1, & \text{if } \rho \text{ is equal to } \rho', \\ 0, & \text{if } \rho \text{ is not equal to } \rho', \end{cases} \qquad \text{for each ray } \rho \text{ of } \Sigma.$$

Conversely, this isomorphism identifies a piecewise linear function $\ell \in \mathrm{PP}(\Sigma)$ on $\Sigma$ with the linear form

$$\ell = \sum_{\rho \text{ ray}} \ell(\mathbf{e}_\rho) x_\rho,$$

and allows us to regard the elements of $A(\Sigma)$ as equivalence classes of piecewise polynomial functions on $\Sigma$, modulo the linear functions on $\Sigma$.

**Example 2.5** (The ring $A(\Sigma_{[0,2]})$) Let us carry out this computation for the braid arrangement $\Sigma_{[0,2]}$, referring to Figure 2. The Courant functions representing the ray variables $x_0, x_1, x_2, x_{01}, x_{02}, x_{12}$ of the previous section are the following, where $t_{ij} := t_i - t_j$:

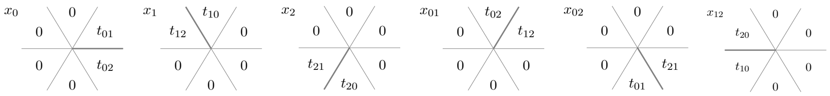
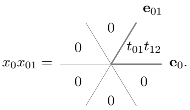


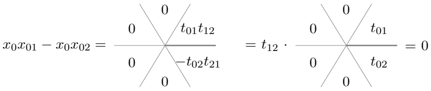

Figure 3: The Courant functions $x_0, x_1, x_2, x_{01}, x_{02}, x_{12}$ on $\Sigma_{[0,2]}$. Each function $x_S$ equals 1 on the marked primitive ray $\mathbf{e}_S$ and 0 on the others.

As we saw in the previous section, $A^0$ is generated by the constant function 1, $A^1$ is generated by $x_0, x_1, x_2, x_{01}$, and $A^2$ is generated by




This expression for the generator $x_0x_{01}$ is supported on the chamber $\mathrm{cone}\{\mathbf{e}_0, \mathbf{e}_{01}\} = \{\mathbf{t} \in \mathrm{N}_E : t_0 > t_1 > t_2\}$. Its unique non-zero polynomial $t_{01}t_{12}$ is the product of the linear forms $t_0 - t_1$ and $t_1 - t_2$ defining the inequalities of the chamber.

It is instructive to double check that two adjacent chambers (and hence any two chambers) give the same generator of $A^2$. The neighbor chamber cone$\{\mathbf{e}_0, \mathbf{e}_{02}\} = \{\mathbf{t} \in \mathrm{N}_E : t_0 > t_2 > t_1\}$ separated by the wall $t_{12} = 0$, gives generator $x_0 x_{02}$. Their difference is




since it is the product of the linear function $t_{12}$ of the wall separating them and a piecewise polynomial function: we have $t_{01} = 0$ on $\mathbf{e}_{01}$, $t_{01} = t_{02}$ on $\mathbf{e}_0$, and $t_{02} = 0$ on $\mathbf{e}_{02}$. This generalizes to any two neighbor chambers in any braid fan, and further, in any simplicial rational fan.

**Computing Degrees** There is a very elegant way to compute the degree of an element $f \in A^n(\Sigma)$ given by a piecewise polynomial $f = (f_\sigma : \sigma \in \Sigma(n))$. To describe it, we first associate a rational function to each facet $\sigma$ of $\Sigma$. If $\sigma$ is simplicial and unimodular, it is generated by $n$ inequalities $f_1(x) \geq 0, \ldots, f_n(x) \geq 0$, where $\{f_1, \ldots, f_n\}$ is the basis dual to the rays generating $\sigma$. This determines a rational function $e_\sigma := 1/(f_1 \cdots f_n)$ in $\mathrm{Sym}^{\pm}(N^\vee)$. In general, we can triangulate $\sigma$ into simplicial unimodular cones $\sigma_1, \ldots, \sigma_n$ and define $e_\sigma := e_{\sigma_1} + \cdots + e_{\sigma_n}$, which turns out to be independent of the triangulation [19]. We then have

$$\deg(f) = \sum_{\sigma \in \Sigma(n)} e_\sigma f_\sigma.$$

It is pleasant and not a priori obvious that this is always a constant, after significant cancellation. It is not so difficult to prove it, though, by verifying that the above formula gives $\deg(x_\sigma) = 1$ for every facet monomial and 0 for every other square-free monomial.

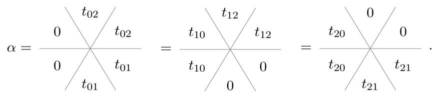
**The Hyperplane and Reciprocal Hyperplane Classes** The elements $\alpha$ and $\beta$ of the permutahedral Chow ring can be described by the following piecewise linear functions, for any $i \in E$:

$$\alpha = \alpha_i = \max(t_i - t_j : j \in E), \qquad \beta = \beta_i = \max(t_j - t_i : j \in E).$$

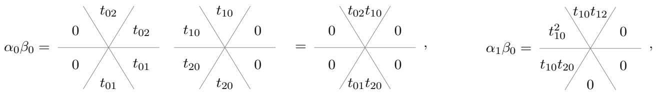
For any $i \neq i'$ the function $\alpha_i - \alpha_{i'} = t_i - t_{i'}$ is linear, and hence in $\mathrm{N}^\vee$, so $\alpha$ is well-defined.[1] To verify the formula for $\alpha_i$, notice that the value of $\max(t_i - t_j : j \in E)$ on $\mathbf{e}_S$ is 1 if $i \in S$ and 0 if $i \notin S$. A similar argument works for $\beta$.

**Example 2.6** (The degree of $\alpha\beta$ in $A(\Sigma_{[0,2]})$.) The special element $\alpha \in A(\Sigma_E)$ is given by the expressions $\alpha_0 = x_0 + x_{01} + x_{02}$, $\alpha_1 = x_1 + x_{01} + x_{12}$, and $\alpha_2 = x_2 + x_{02} + x_{12}$, which give:

[1]It is tempting but incorrect to think that $t_i$ is linear so we can write $\alpha = max(-t_j : j \in E)$: in fact $t_i$ is not even a well defined function on the ambient space $\mathrm{N} = \mathbb{R}^{[0,2]} / \mathbb{R}\,\mathbf{e}_{[0,2]}$.




These look different, but they are equal modulo global linear functions on $N$: the first two differ by $t_{01}$ and the latter two differ by $t_{12}$. Similarly, there are three natural piecewise linear representatives for $\beta$, namely $\beta_0, \beta_1, \beta_2$. Let's compute the degree of $\alpha\beta$ in two ways, referring to Figure 2 again. Since




we have that $\deg(\alpha\beta)$ equals

$$\frac{t_{02}t_{10}}{t_{02}t_{10}} + \frac{t_{01}t_{20}}{t_{01}t_{20}} = 2 \text{ and } \qquad \frac{t_{10}t_{12}}{t_{02}t_{10}} + \frac{t_{10}^2}{t_{12}t_{20}} + \frac{t_{10}t_{20}}{t_{21}t_{10}} = 2,$$

where the first computation is immediate and the second involves a fun cancellation.

## 2.3 The Chow Ring in Terms of Minkowski Weights

**The Chow Ring** A *$k$-dimensional Minkowski weight* on $\Sigma$ is a real-valued function $\omega$ on the set $\Sigma(k)$ of $k$-dimensional cones that satisfies the *balancing condition*: For every $(k-1)$-dimensional cone $\tau$ in $\Sigma$,

$$\sum_{\tau \subset \sigma} \omega(\sigma)\mathbf{e}_{\sigma/\tau} = 0 \text{ in the quotient space } \mathrm{N} / \operatorname{span}(\tau),$$

where $\mathbf{e}_{\sigma/\tau}$ is the primitive generator of the ray $(\sigma + \operatorname{span}(\tau))/\operatorname{span}(\tau)$. We say that $w$ is *positive* if $w(\sigma)$ is positive for every $\sigma$ in $\Sigma(k)$. We write $\mathrm{MW}_k(\Sigma)$ for the space of $k$-dimensional Minkowski weights on $\Sigma$, and set $\mathrm{MW}(\Sigma) = \bigoplus_{k \geq 0} \mathrm{MW}_k(\Sigma)$.

The product in $\mathrm{MW}(\Sigma)$ is given by the following *fan displacement rule*. If $X_1$ and $X_2$ are Minkowski weights of codimension $k$ and $\ell$ on $\Sigma$, then their product is defined to be the *stable intersection*

$$X_1 \cdot X_2 := \lim_{\epsilon \to 0} X_1 \cdot (X_2 + \epsilon\,\mathbf{v})$$

for any vector $\mathbf{v} \in \mathrm{N}$ such that $X_1$ and $X_2 + \epsilon\,\mathbf{v}$ intersect transversally for sufficiently small $\epsilon > 0$. The facets of $X_1 \cdot X_2$ are the $(k+\ell)$–codimensional intersections of a facet of $X_1$ and a facet of $X_2$. The weight of a facet $\tau$ of $X_1 \cdot X_2$ is

$$w(\tau) = \sum_{\sigma_1, \sigma_2} w(\sigma_1)w(\sigma_2)[\mathbb{Z}^n : L_{\mathbb{Z}}(\sigma_1) + L_{\mathbb{Z}}(\sigma_2)],$$

summing over the facets $\sigma_1$ and $\sigma_2$ of $X_1$ and $X_2$ respectively such that $\tau = \sigma_1 \cap \sigma_2$ and $\sigma_1 \cap (\sigma_2 + \epsilon\,\mathbf{v}) \neq 0$ for small $\epsilon > 0$. It is non-trivial that the construction above is independent of the choice of a (generic) vector $\mathbf{v}$, and that it is also a Minkowski weight, that is, it satisfies the balancing condition [28, 35].

When $\Sigma$ is complete, Fulton and Sturmfels [28] proved that

$$A(\Sigma) \cong \mathrm{MW}(\Sigma)$$

so understanding the Chow ring of $\Sigma$ is equivalent to understanding Minkowski weights on $\Sigma$ and their stable intersections.

**The Dictionary** For $\Sigma$ complete, Katz and Payne [37] described the canonical[2] map from $\mathrm{PP}(\Sigma)$ to $\mathrm{MW}(\Sigma)$ that descends to an isomorphism $A^k(\Sigma) \cong \mathrm{MW}_{n-k}(\Sigma)$. We focus on a different description for a special case: when $f \in \mathrm{PP}^1(\Sigma)$ is a piecewise linear function that is convex, that is, $f((\mathbf{x}+\mathbf{y})/2) \leq (f(\mathbf{x})+f(\mathbf{y}))/2$ for all $\mathbf{x}, \mathbf{y} \in \mathrm{N}$. In this case, $f$ can be written as a *tropical polynomial*; that is, the maximum of a finite number of linear functions:

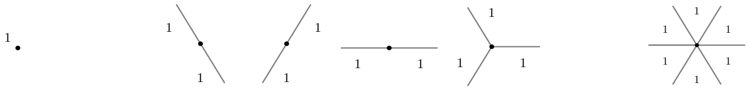
$$f(\mathbf{x}) = \max\{\mathbf{v}_1(\mathbf{x}), \ldots, \mathbf{v}_m(\mathbf{x})\} \qquad \text{for } \mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathrm{N}^\vee .$$

The *corner locus*, where this function is not linear, is the *tropical hypersurface*:

$$\mathrm{trop}\, f = \{\mathbf{x} \in \mathrm{N} : \max_{1 \leq i \leq m} \{\mathbf{v}_i(\mathbf{x})\} \text{ is achieved at least twice}\}.$$

This is the $(n-1)$-skeleton of the normal fan of the *Newton polytope* $\mathrm{Newt}(f) = \mathrm{conv}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$. It turns into a balanced fan with a natural choice of weights: for each facet $F$ of $\mathrm{trop}\, f$ the weight $w(F) = \ell(F^\vee)$ equals the lattice length of the corresponding edge of $\mathrm{Newt}(f)$. This balanced fan is the Minkowski weight in $\mathrm{MW}_{n-1}(\Sigma)$ corresponding to $f$. For details, see [40, 41, 43].

**Example 2.7** (The Chow ring $A(\Sigma_{[0,2]})$.) Let $\Sigma = \Sigma[0,2]$. For $k=0$ the balancing condition is vacuous and a Minkowski weight is a choice of a weight on the origin. For $k=1$, we need to put a weight on each of the six rays so that the weighted sum of the rays is 0. The four choices of weight below generate all others. For $k=2$ we need weights on each maximal cone of $\Sigma$. Each ray $\tau$ is in two cones $\sigma_1$ and $\sigma_2$ which satisfy $\mathbf{e}_{\sigma_1/\tau} = -\mathbf{e}_{\sigma_2/\tau}$, so the balancing condition says that $w(\sigma_1) = w(\sigma_2)$, and hence all weights are equal. Thus $\mathrm{MW}(\Sigma)$ is spanned by the following Minkowski weights:




**Computing Degrees** One can use the *fan displacement rule* to compute the degree of a product: $X_1 \cdot X_2 = \lim_{\epsilon \to 0} X_1 \cdot (X_2 + \epsilon\, \mathbf{v})$, where $\mathbf{v} \in \mathrm{N}$ is any vector such that $X_1$ and $X_2 + \epsilon\, \mathbf{v}$ intersect transversally for sufficiently small $\epsilon > 0$. This requires one to understand how these fans intersect by solving systems of linear equations and inequalities. Sometimes a clever choice of $\mathbf{v}$ – for example one whose coordinates increase very quickly – can simplify the computations.

[2]This is canonical in the sense that $\mathrm{PP}(\Sigma) \cong A_T(X_\Sigma)$ and $\mathrm{MW}(\Sigma) \cong A(X_\Sigma)$ are isomorphic to the equivariant and the ordinary Chow cohomology rings of the toric variety $X_\Sigma$, respectively, and there is a canonical map $A_T(X_\Sigma) \to A(X_\Sigma)$.