978-1-009-47938-7 — Introducing the Philosophy of Mathematical Practice
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Excerpt
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Introduction

Mathematics is a fascinating yet mysterious field. Some are drawn to it because of its rigour and because it yields absolute truths. Others are taken by mathematics' capacity to describe our complex world in simple relations. When engaging with mathematics, however, questions arise. One might wonder how we were even able to develop and grasp all of the complex structures that constitute contemporary mathematics. A related question concerns the nature of the relations between elementary mathematical activities such as counting and measuring and the modern abstract theories of mathematics – if there are any such relations. Mathematics has a language of its own with technical terms, notations that are used to write expressions in a compact and convenient form, and a variety of visual representations such as figures and diagrams. This special language is also part of what makes mathematics difficult to understand. In this light, one may ask: What is the role of the notation and visual representations – are they essential components of contemporary mathematics or merely useful devices? Could it also be that they somehow aid our discovery of mathematical facts? These are examples of questions that are of interest to the philosopher of mathematical practice and some of them will be addressed in this Element.

The main purpose of this Element is to introduce the field of philosophy that takes as its starting point the practice of mathematics. This is a difficult task – in part since the field is still young. This means that, even though there is some consensus as to what the philosophy of mathematical practice (abbreviated as PMP) may designate, scholars claiming to be in the field still characterise it in quite distinct ways. One way to describe PMP is to contrast it with mainstream philosophy of mathematics. The following four general trends give a rough idea. First, PMP aims to extend the topics that can be studied as part of the philosophy of mathematics beyond the traditional questions concerning the foundations and ontology of mathematics. Second, philosophers of practice often take interest in more specific questions, such as 'What is a mathematical explanation?', in contrast to the broader pictures that one finds in philosophy. Third, methods drawn upon may be different. Philosophy of mathematical practice is historically sensitive, analyses case studies, and sometimes even refers to scientific or empirical results whereas mainstream philosophy of mathematics belongs to analytic philosophy and develops and is based on formal tools. Finally, mathematics is often portrayed as static, a collection of eternal truths, in mainstream philosophy. In PMP, we are (also) interested in mathematics' dynamic character, how concepts are defined, proofs are found, and so on, stressing that mathematics is an activity done by human agents. This *is* a rough sketch and does not capture the richness of the field. Philosophy of mathematical practice, for example, also draws on analytic tools and in certain cases seeks to formulate 'big pictures'.

The Element comprises three sections. Each section can be read independently, but there are examples that are referred to across the sections. The first section introduces the 'philosophy of mathematical practice' from a general point of view. The section provides an outline of some of the different approaches to PMP mainly through an analysis of what 'practice' refers to.

In addition to PMP, scholars have also used 'mathematical philosophy' to characterise the type of philosophy of mathematics that interacts with mathematical practice proper. This term was used by Bertrand Russell about a century ago and refers to the general movement, scientific philosophy, which was active between 1850 and 1930. There are interesting similarities (as well as differences) between this movement and the current PMP, some of which will be pointed out.

The ensuing two sections treat in more detail two examples of work in the philosophy of mathematical practice. The two topics are Mathematical Structuralism and Visual Thinking in Mathematics. One motivation to adopt a structuralist position stems from developments within mathematics itself. Structuralism therefore is an exemplary case of a philosophical position inspired by mathematical practice. Emphasising a dynamic conception of mathematics, I focus on methodological aspects of structuralism. This includes using the axiomatic method as a tool to create structures and to organise mathematics in various ways. Furthermore, I consider the role of relations in mathematics on a 'global' scale which entails understanding mathematical structuralism in a broader sense than is typical in the philosophy of mathematics.

Section 3 contains a brief historical introduction to the use of diagrams in mathematics, starting with the observation that diagrams formed an essential component of Greek mathematics. Euclid's *Elements* was long regarded as the paradigm for how mathematics should be studied and presented, and so geometry served for many centuries as a foundation for mathematics. However, during the eighteenth century, this picture gradually shifted, and at the turn of the twentieth century one finds explicit statements that proofs should not rely on geometric intuition nor be based on diagrams. In the latter part of the twentieth century, philosophers contested this view, noticing the prevalence of diagrams in mathematical practice. Consequently, scholars began investigating the role that figures, or diagrams, play. We will consider some contributions that examine the use of diagrams in proofs, noting that each is based on a careful analysis of exemplary cases and that diagram-based reasoning can be supported by formal arguments. Finally, I will highlight some of the advantages that diagrams offer over other types of representations.

1 Different Approaches to Philosophy of Mathematics

There are different ways in which mathematics presents itself and gives rise to philosophical reflections. One might think of mathematics (for example, at a certain point in time) as a collection of theories that establish relations between concepts, formulated in terms of propositions, and that we have knowledge of a proposition in case there is a proof of it. Questions that this picture might raise concern the nature of the involved mathematical concepts, how to account for the apparent necessity of mathematical propositions, and which requirements should be stated for arguments to be accepted as rigorous proofs. This is a static conception of mathematics. In contrast, one might be interested in mathematics as an activity wondering, for example, how concepts are developed and theories formed. One might also enquire about various epistemic concerns such as why mathematicians value multiple proofs of the same theorem. One might think that this division – between a static and dynamic conception of mathematics – is simple and that the questions they give rise to as well as the methods employed to respond to them are disjoint. This turns out not to be true, as we shall see.

The label ‘The philosophy of mathematical practice’ (PMP) will be reserved for topics within the philosophy of mathematics that explicitly address themes that are tied to the practice of mathematics, interests of mathematicians (present and past) that are related to their mathematical practice, or challenges posed by the content of mathematics (broadly construed). ‘Practice’ may, but need not, imply an underlying assumption that mathematics is done by human agents. There are a number of different approaches to the philosophy of mathematics that take an ‘agent based’ perspective, that is, perspectives based on the fact that mathematics is done by human beings or is the outcome of human activities. Such a point of view affects, among other things, how knowledge is characterised. I return to this point in Section 1.2. Other approaches to PMP insist that philosophical reflections are informed by mathematics itself, for example, mathematical theories, concepts, how proofs convince, are found or presented. Furthermore, ‘philosophy that is informed by mathematics’ could also mean that mathematics itself provides the tools to solve philosophical problems. It may not be entirely clear what I mean by these brief descriptions, but the current and two consecutive sections are intended to give some concrete examples that illustrate these points.

One could also ask how the ‘philosophy of mathematical practice’ differs from the philosophy of mathematics. This is, indeed, a relevant, but difficult, question. One thing that makes it complicated is the many different traditions (both past and present) of thinking about mathematics. In light of this, ‘the philosophy of mathematics’ cannot refer to a single approach. This entails that what I will be characterising as the 'philosophy of mathematical practice' may be compatible with both past and existing traditions. Aldo Antonelli (2001), characterising what he refers to as 'mathematical philosophy', makes a related point when noting that mathematical philosophy has been practised by a number of past thinkers from Plato to Hilbert. Antonelli contrasts mathematical philosophy to contributions that are exclusively concerned with ontological questions, 'the epistemology of mathematical propositions, or the necessary status of mathematical truths' (2001, p. 1) – questions that elsewhere are claimed to belong to mainstream philosophy of mathematics.

Early proponents of a 'practice oriented' approach objected to the one-sided focus on foundational questions and the use of formal tools, namely formal logic and set theory, when dealing with them. Some of them, referred to as 'mavericks', even claimed that mathematics does not need foundations; see Mancosu (2008a) for an elaboration. Less radical philosophers have instead urged that the range of topics and questions considered should be extended. A further requirement is that answers to these questions should draw on relevant mathematical theories (that go beyond arithmetic and set theory).

In this first section I introduce the philosophy of mathematical practice (PMP) and mathematical philosophy that is related to it. 'Mathematical philosophy' has recently been used to characterise philosophical work that employs mathematical tools to address problems in mathematics (Weber 2013) and, as mentioned earlier, to offer a particular perspective on the philosophy of mathematics (Antonelli 2001). The label was coined by Bertrand Russell (1919) and appears in the title of his book *Introduction to Mathematical Philosophy*.[1] Russell's 'mathematical philosophy' can be seen as part of the general movement 'scientific philosophy' which started in the mid nineteenth century. Through the influence of Russell and others, it transformed into what is today known as analytic philosophy.[2] Mathematical philosophy as characterised by Antonelli can be seen as a version of PMP. It may therefore seem odd that it can be traced back to Russell who has been highly influential in the development of analytic philosophy and is one of the fathers of the logicist programme, one of the foundational schools. Both the foundational schools and formal tools used in analytic philosophy, or at least certain versions of these, have been criticised by

[1] 'Mathematical philosophy' has a number of different interpretations. It could mean the use of mathematical, or formal tools in philosophy (cf. Munich Center for Mathematical Philosophy), that views on mathematics inform one's philosophy (as in the writings of C.S. Peirce), or as a view on the philosophy of mathematics that takes mathematical practice seriously.

[2] Some of the early philosophers usually mentioned as the originators of analytic philosophy are Moore, Russell, and Wittgenstein. Wittgenstein was in particular influenced by Russell and Frege.

the mavericks and other precursors to PMP. If one considers the original motivation and general ideas of scientific or mathematical philosophy, however, it turns out that they resonate with ideas of PMP. This need not imply that the particular outcomes, that is, the positions of these orientations have to overlap and that there is agreement on every point. Not everyone agrees with Russell, for example, that mathematics is the same as formal logic (Russell 1901).

Section 1.1 briefly introduces mathematical philosophy as it was conceived by Russell and the context of scientific philosophy. The main point is to illustrate that scientific philosophy and Russell's mathematical philosophy contain certain ideas that are shared with PMP. Another main point will be to illustrate that what the philosophy of mathematics is, which topics are of interest to philosophers, and what methods are used change over time. Section 1.2 discusses general themes of the philosophy of mathematical practice.

1.1 Scientific and Mathematical Philosophy

In the introduction to a special issue of *Topoi*, Antonelli defines 'mathematical philosophy' as *that area of philosophical reflection that is contiguous to, and interactive with, mathematical practice proper* (Antonelli 2001, p. 1). To be interactive with mathematical practice entails that one draws on relevant parts of mathematics: *Paraphrasing Kant, one could say that mathematics without philosophy is blind, and philosophy without mathematics is empty* (Antonelli 2001, p. 1).[3] Antonelli further claims that philosophy that is not informed by mathematical practice risks becoming either legislative or apologetic. In contrast, mathematical philosophy 'is respectful of, but not subsidiary to, current mathematical practice. It engages the issues, points out conceptual tensions, and highlights unexpected consequences. Mathematical philosophy positions itself neither above nor below mathematics, but rather on a par with it, taking the role of an equal interlocutor' (Antonelli 2001, p. 1). Antonelli attributes the term 'mathematical philosophy' to Russell, to whom we now turn.

Mathematical philosophy is part of scientific philosophy that arose around the mid nineteenth century as a reaction against the post-Kantian German idealism of, for example, Hegel and Schelling and, in the case of Russell, also nineteenth-century British idealism.[4] The scientific philosophers wished instead to base philosophy on the methods of science that had proven far more successful. The former grand systems of philosophy were regarded as individualistic and subjective in contrast to the scientific methods that were taken to be progressive, collaborative, and objective.

[3] See Kant's *Critique of Pure Reason*, A51/B51.
[4] Further details can be found in Richardson (1997) and Preston (n.d.).

The label ‘scientific philosophy’ was used by Hermann von Helmholtz in a famous talk celebrating Kant in 1855 in Königsberg on the occasion of the dedication of a monument to Kant. In this talk, Helmholtz noted the enmity and distrust between science and philosophy. He urged instead that they should collaborate. According to Helmholtz, philosophers should turn to the theory of knowledge and base their theories on the recent developments of relevant fields (i.e., psychology and physiology) instead of building grand systems of metaphysics. The list of scientific philosophers is long and among twentieth-century philosophers we find Schlick, Carnap, and Quine (Friedman 2012), Russell, Husserl, and the early Heidegger (Richardson 1997).

It is clear from this list that ‘scientific philosophy’ covers quite diverse approaches to philosophy. What makes them ‘scientific’ is summed up by Heidegger in a lecture given in 1925 (quoted from Richardson (1997), p. 441):

1. Because it is a philosophy of the sciences, that is, because it is a theory of scientific knowledge, because it has as its actual object the fact of science.
2. Because by way of this inquiry into the structure of already given sciences it secures its own theme that it investigates in accordance with its own method, while it itself no longer lapses into the domain of reflection characteristic of the particular sciences. It is “scientific” because it acquires its own domain and its own method. At the same time, the method maintains its security by its constant orientation to the factual conduct of the sciences themselves. Speculation aimed at world views is thereby avoided.[5]

While the scientific philosophers shared a common enemy in the former ‘speculative world views’, there was little agreement on which part of science or scientific method should replace them. According to Richardson, Russell has to a large extent influenced how scientific philosophy is understood today, that ‘scientific’ means using the relevant logical tools (p. 424). He notes, however, that quite different interpretations existed. Husserl, for example, also regarded himself as a scientific philosopher (and is later mentioned by Heidegger as belonging to that tradition). To Husserl, the scientific method used in his phenomenology consisted in analysing ‘pure consciousness’. In relation to mathematics, one might also take the point of view that ‘scientific’ refers to mathematics itself in the sense that mathematical or meta-mathematical tools are drawn on when solving philosophical problems. Referring to Hilbert’s

[5] I omit Heidegger’s third point, which places the scientific philosophers in the tradition of phenomenology, or the science of consciousness.

contributions to meta-mathematics, Hourya Benis-Sinaceur (2018) argues that Hilbert can be considered as a scientific philosopher in this latter sense.[6]

An important part of the general scientific method, and an idea that is shared among the scientific philosophers, is that scientific questions are solved by collaborative efforts and that this allows researchers to split big problems into smaller and manageable ones (Richardson, p. 434). Here formulated by Russell:

> It is chiefly owing to this fact that philosophy, unlike science, has hitherto been unprogressive, because each original philosopher has had to begin the work again from the beginning, without being able to accept anything definite from the work of his predecessors. A scientific philosophy such as I wish to recommend will be piecemeal and tentative like other sciences … What is feasible is the understanding of general forms, and the division of general problems into a number of separate and less baffling questions. "Divide and conquer" is the maxim of success here as elsewhere. (Russell 1914, p. 113)

In contrast to scientific investigations, however, the scientific philosopher must not rely on empirical facts:

> [a] philosophical proposition must be such as can be neither proved nor disproved by empirical evidence. Too often we find in philosophical books arguments based upon the course of history, or the convolutions of the brain, or the eyes of shell-fish. Special and accidental facts of this kind are irrelevant to philosophy, which must make only such assertions as would be equally true however the actual world were constituted. (Russell 1914, p. 111)

Philosophy should be based on general propositions (which is possible with the help from logic) that are a priori.

An important component of Russell's mathematical philosophy was his use of the recently developed tools from logic, most notably by Gottlob Frege. Frege introduced quantificational logic which, in addition to quantifiers, employs the concept of a function that can be applied to one or multiple arguments. Whereas propositions were earlier analysed in the form 'predicate and subject', Frege's logic allows one to use the components 'A(d)' or 'A(x)' where A stands for some property and 'd' an individual to which the property

6 Benis-Sinaceur (2018) focuses on Hilbert's use of the Kantian notion of critique and explains in detail how Hilbert puts his axiomatic method and proof theory under the banner of critique (referring to, e.g., Hilbert's *Grundlagen der Geometrie* 1899, and 'Axiomatic Thought' 1918). Hilbert refers to the investigation of the foundation and logical structures of geometry in his *Foundations of Geometry* as 'an analysis of the intuition of space' (p. 27). In contrast to Kant's view, Hilbert's 'intuition' is objective (and associated with his 'finitist attitude'): '*intuition is rooted in perceiving sensory signs outside of the mind*'. Put briefly, Hilbert solves the problems of mathematics by drawing on meta-mathematical tools, replacing the role of philosophy with mathematics itself (Benis-Sinaceur 2018, p. 35).

is ascribed. In this way, it is possible to capture the logical structure of sentences of the form ‘For all x P(x)’ or ‘There exists an x such that P(x)’ for some predicate P (and, of course, much more complicated sentences using nested quantification and relational symbols). The new logic was introduced as part of Frege’s logicist programme that intended to demonstrate that the theory of numbers could be reduced to logic. It is well-known that Frege’s original attempt failed: Russell discovered that one of Frege’s assumptions gave rise to the paradox known as Russell’s paradox. The assumption allows one to consider the extension of any conceivable concept.[7] If one considers the property of ‘not belonging to itself’, it is possible to form ‘the set of all sets not being a member of themselves’, which gives rise to the paradox. Russell, being convinced of the overall correctness of Frege’s programme, was not discouraged by it. Broadening the project, Russell believed that all of pure mathematics belonged to logic and set himself the task to demonstrate that this was the case. This was carried out in collaboration with A. N. Whitehead and led to the monumental work *Principia Mathematica* published in 1910–1913.

Coming back to the topic of mathematical philosophy, the quantificational logic is an important tool in one of Russell’s most important contributions to analytic philosophy, the theory of descriptions. Put briefly, the theory of descriptions uses logical analysis of propositions to dissolve philosophical problems. To see how it is used consider the statement ‘The present king of France is bald’. This statement is intuitively false since there is no king of France. On previous interpretations (the object theory of meaning), though, one could only claim that it is false if one allows that there is a universe of non-empirical objects that contains the king of France. On Russell’s interpretation, the problem is dissolved once one finds the statement’s correct logical form: it can be reformulated as ‘There is an x and x is king of France and x is bald’. Since there is no individual, x, for which ‘x is king of France’ is true, the statement as a whole is false.[8]

In an early paper, Russell (1901) used the label ‘mathematical philosophy’ in a broader sense, namely, that philosophy should pay attention to the results and methods of mathematics when solving philosophical problems. Problems may arise because of a confusing conception of a concept. The method is conceptual analysis. A concept that has given rise to numerous puzzles is the ‘infinite’. Zeno’s paradox, for example, demonstrates that motion is an illusion: for motion to be possible, one has to move across a distance consisting of

[7] The extension of a concept, say F, is all objects, a, for which the statement ‘a is F’ is true. In formal terms it can be interpreted as the collection $\{x: Fx\}$.
[8] I refer to Russell’s *Introduction to Mathematical Philosophy* for further details.

an infinite number of parts in a finite amount of time which seems impossible. Russell credits Weierstrass, Dedekind, and Cantor for having found precise definitions of the infinite. Using the new theories of the infinite, the infinitely small and large and sums of infinite sequences, Russell claims, it is possible to find solutions to the philosophical paradoxes. A mathematical solution to Zeno's paradox exploits the fact that infinite sums may converge. The particular sum that is used in Zeno's paradox is $\sum_{i=1}^{\infty}(\frac{1}{2})^i$, that converges to 1.

The infinitely small posed another challenge to mathematicians in the form of infinitesimals until Cauchy introduced the now well-known notion of a limit in the beginning of the nineteenth century. Cauchy further formulated a theory of infinite series drawing the important distinction between a convergent and a divergent series, formulating criteria of convergence and introducing the concept of a radius of convergence. When referring to the infinitely large, Cantor's contributions are most often mentioned: in particular his introduction of cardinal and ordinal numbers. Although many problems concerning the infinite thus found solutions at the turn of the twentieth century, mathematicians and philosophers still discuss the nature of the infinite in a number of contexts. See Easwaran et al. (2023) for an overview. We return to the question of how to characterise the infinitely large at the end of this section.

Before turning to the philosophy of mathematical practice, I note some of the ideas that philosophers of mathematical practice might share with scientific philosophy. The first point concerns the relations between philosophy and mathematics. Russell's advice to collaborate and establish relations between philosophers and mathematicians is still relevant today. Collaborations exist but are rare. Philosophy of mathematical practice further agrees with the scientific philosophers that philosophy should look to recent or relevant parts of mathematics when finding solutions to philosophical problems. Solutions to problems may draw on mathematical results or tools as in Hilbert's version of scientific philosophy. We find examples of this in the following sections. At the same time, we should keep in mind that philosophy and mathematics are two different domains with distinct subject matter and methods. Only in this way is it possible that philosophy can play the role of an equal interlocutor and be a useful guide for mathematics as well as the converse.

The scientific philosopher's guiding principle of 'divide and conquer' fits well with PMP that often asks more specific questions related to the practice of mathematics. One difference between Russell's mathematical philosophy and that of some contemporary philosophers of mathematics consists in the extent of empirical claims or scientific results that can be drawn upon. It is possible to find contemporary philosophers who note that a particular proof is claimed to be beautiful or explanatory and then set themselves the task to formulate a philosophical account of aesthetic judgements that explains the claim. Similarly, a philosopher may draw on cognitive science and theories of perception when explaining how mathematical knowledge is acquired.

Following the scientific philosophers' advice to pay attention to the methods of mathematics, however, does not entail that we have to agree with the particular positions formulated by the scientific or mathematical philosophers. The challenges and concerns of contemporary mathematics are not the same as at the end of the nineteenth century. Indeed, as mathematics develops so do its methods and issues. Most contemporary mathematicians do not seem very interested in the foundations of mathematics as it was conceived around the turn of the nineteenth century. It is, of course, still possible to find mathematicians that are worried about the unrestricted use of the infinite (e.g., using various forms of the axiom of choice), and certain parts of analysis depend on deep set theoretical results. One might also note the recent development of proof assistants that has revived the interest in formal proofs of mathematics (see Avigad (2021)). Besides questions on the foundations of mathematics, a number of other topics are discussed. In light of the growing diversity of mathematical disciplines, one might ask what unifies mathematics. This question was already posed during the twentieth century (Bourbaki 1950) and is still relevant today. Another concern comes from the increased use of computers in mathematics. Besides the use of formal proof checkers, computers have revolutionised how mathematics is done, or at least provided mathematicians with different types of, and in some sense, much stronger tools. Computers are used to experiment (in different senses of the word), to verify (in some cases), to write papers, and to communicate with peers (emails, talks) and to access and store material.

1.2 Philosophy of Mathematical Practice

The label 'philosophy of mathematical practice' has been applied to a number of different approaches to studies of mathematics. The Association for the Philosophy of Mathematical Practice characterises them in the following broad sense:

> Such approaches include the study of a wide variety of issues concerned with the way mathematics is done, evaluated, and applied, and in addition, or in connection therewith, with historical episodes or traditions, applications, educational problems, cognitive questions, etc. We suggest using the label