1 Introduction and Technical Preliminaries

1.1 Introduction

Evolutionary games form a class of noncooperative games in which the interaction of strategies is studied using evolutionary ideas from two different approaches, static and dynamic. The static approach captures evolutionary concepts through defining and studying equilibrium terms. The dynamic approach, on the other hand, studies the interaction of strategies as a dynamical process determined by a system of differential equations. This Element concerns the dynamic approach with a specific dynamical system known as the replicator dynamics. We are particularly interested in the stability of the replicator dynamics for evolutionary games in which the strategy set is a measurable set or, more precisely, a separable metric space.

An evolutionary game is said to be *symmetric* if there are two players only and, furthermore, they have the same strategy sets and the same payoff functions. This type of game models interactions of strategies of a single population and forms part of the so-called population games. On the other hand, *asymmetric* evolutionary games, also known as *multipopulation* games, are those in which there is a finite set of players (or populations), each of which has a different set of strategies and different payoff functions.

Game models with strategies in general measurable spaces are important because they include essentially all the models that appear in theory and applications, from games with finite strategy sets to games with strategies in metric spaces such as some models in oligopoly theory, international trade theory, war of attrition, and public goods, among others. With our proposed model we can introduce evolutionary dynamics in games where the strategy set is a Borel space (that is, a Borel subset of a complete and separable metric space). We have, consequently, that the dynamical system lives in a Banach space, which in our case is a space of finite signed measures. In particular, if the strategy set is finite, then the dynamical system is in $\mathbb{R}^m$, where m is the number of strategies of a player for symmetric games, or the total number of strategies of all players for asymmetric games.

The main objective of this Element is to present a general, unified framework to study the existence of solutions and the stability of the replicator dynamics for games with metric strategy sets. This means that, first, we establish conditions for the existence of solutions to the replicator dynamics for asymmetric games, and also conditions that ensure the stability of the system in this asymmetric case. Bomze and Pötscher (1989) suggest an approach (similar to that in Selten (1980) for the case with finite strategy sets) in which asymmetric games are reduced to the symmetric case; for details, see Section 3.2. This approach, however, has some disadvantages. For instance, the relationship between Nash equilibria and the replicator dynamics is unclear. It is also unclear how to extend stability concepts and results to the asymmetric case. In contrast, with our proposed model it is easy to see the relationship between a Nash equilibrium and the replicator dynamics (see Section 3.4) and, in addition, stability concepts have a natural extension from the symmetric case to the asymmetric situation (see Section 3.5).

Second, for symmetric games we study stability criteria in a fairly general context, with respect to different topologies and metrics on a space of measures. We can thus, for instance, relate the Nash equilibria of a certain normal-form game with the stability of the replicator dynamics under different metrics (see Section 4.4), and similarly for strongly uninvadable strategies (Section 4.3), a refinement of Nash equilibria.

Third, we can obtain quite general, and at the same time precise results on the approximation of the replicator dynamics by different approximating models, which include finite-dimensional dynamic systems. These approximations can be in a strong form, using the variational norm (Section 5.2), or in a weak form, using the Kantorovich–Rubinstein norm (Section 5.3).

Fourth, we study the replicator dynamics as a limit of a sequence of Markov processes (see Section 6), where each Markov process describes a stochastic interaction among the characteristics (genotypes or actions) of the individuals. This stochastic interaction can be studied by means of a determinist dynamic under some hypotheses.

Concerning some related literature, conditions for the existence of solutions to the replicator dynamics in measure spaces in the symmetric case are given by several authors, including Bomze (1991), Oechssler and Riedel (2001), and more generally (including dynamics different from the replicator equation) by Cleveland and Ackleh (2013). In Section 3.3 we present conditions for the existence of solutions to the replicator dynamics in measure spaces in the asymmetric case and some other important results.

Similarly, conditions for stability have been developed with respect to different topologies, as in, for instance, Bomze (1990), Oechssler and Riedel (2001, 2002), Eshel and Sansone (2003), Van Veelen and Spreij (2009), and Cressman and Hofbauer (2005). In Section 3.5, we present stability results for the replicator dynamics in the asymmetric case. In Section 4.3, we present a brief review of stability results in the symmetric case. Also in Section 4.3 we establish a result that characterizes the stability of the replicator dynamics with respect to the Wasserstein metric, which is analogous to theorem 2 of Bomze (1990) and is also an approximation to answer a conjecture proposed by Oechssler and Riedel (2002).

An important issue in evolutionary games is to study the replicator dynamics as a limit of a sequence of Markov processes describing interactions among individuals in a population. These stochastic interactions describe the evolution of the species. There are many references on this issue when the strategy space is finite; for instance, to name a few, Benaim and Weibull (2003), Corradi and Sarin (2000), and Sandholm (2003, 2010). However, a more general mathematical structure is needed if the strategy set is a measurable space, which is what we propose in Section 6.

On the other hand, in the theory of evolutionary games there are several interesting dynamics, such as the imitation dynamics, the monotone-selection dynamics, the best-response dynamics, the Brown–von Neumann–Nash dynamics, and so forth (see Hofbauer and Sigmund (1998), Hofbauer and Sigmund (2003), Sandholm (2010), among others). Some of these evolutionary dynamics have been extended to games with strategies in a space of probability measures. For instance, Hofbauer et al. (2009) extend the Brown–von Neumann–Nash dynamics; Lahkar and Riedel (2015) extend the logit dynamics; and Lahkar et al. (2022) develop other related results. Moreover, Cheung (2014, 2016) proposes a general theory for pairwise comparison dynamics and for imitative dynamics. Ruijgrok and Ruijgrok (2015) extend the replicator dynamics with a mutation term. For asymmetric games, Mendoza-Palacios and Hernández-Lerma (2015) establish conditions of existence; in addition, Mendoza-Palacios and Hernández-Lerma (2015) and Narang and Shaiju (2019, 2020, 2022) define static equilibria and analyze conditions for the stability of the replicator dynamics in asymmetric games.

Among all these dynamics, we selected the replicator dynamics partly because it is the most studied for games with strategies in metric spaces, and partly because it has many interesting properties, as can be seen in Cressman (1997), Hofbauer and Weibull (1996), and many other references. In particular, with the replicator dynamics it is not difficult to construct a proof of the existence of Nash equilibria and, moreover, when the strategy set is finite, we can give a geometric characterization of the set of Nash equilibria; see Harsanyi (1973), Hofbauer and Sigmund (1998), and Ritzberger (1994).

Finally, it is noteworthy that today evolutionary games have many applications in different areas. For example, genetics and biology, Hofbauer and Sigmund (1998); modeling cancer, Gatenby and Gillies (2008), Hummert et al. (2014), Vincent and Gatenby (2005); and spread of epidemics, Bauch (2005), Yang and Yang (2015). In economics and social sciences, there are applications in different areas: in criminal behavior, Cressman et al. (1998), Quinteros and Villena (2022); corruption, Kou et al. (2021), Katsikas et al. (2016); forest management and economy of natural resources, Sethi and Somanathan (1996), Shahi and Kant (2007), Lamantia (2017); spatial economy and economic development, Fujita et al. (2001), Accinelli and Sánchez Carrera (2012), Araujo and de Souza (2010); combatting money laundering and finance, Araujo (2010), Amir et al. (2011), Amir et al. (2013); industrial organization, Bischi et al. (2015), Almudi et al. (2020c); industrial policy and innovation economics, Almudi and Fatas-Villafranca (2021), Almudi et al. (2020a), Mendoza-Palacios et al. (2022), Mendoza-Palacios and Mercado (2021), among others. We selected the examples in Section 2 because most of them are classical models in the literature of game theory and, in addition, the corresponding strategy sets are metric spaces. This allows us to relate some of our main theoretical results to interesting particular applications.

1.2 Summary

The remainder of this Element is organized as follows. Section 1.3 presents notation and technical requirements.

Section 2 introduces a normal-form game and presents important related concepts. We also show examples that will be used in the rest of this Element.

Section 3 introduces an evolutionary dynamics for asymmetric games. Section 3.1 shows a heuristic approximation to the replicator dynamics for the asymmetric case. Section 3.2 describes the asymmetric evolutionary game and the replicator dynamics. Section 3.3 establishes conditions for the existence of a solution to the system of differential equations describing the replicator dynamics, and gives some characterizations of the solution. Section 3.4 establishes a relationship between *Nash equilibria* and the replicator dynamics. Section 3.5 introduces conditions to establish the stability of the replicator equations. Section 3.6 proposes examples to illustrate our results. We conclude the section in Section 3.7 with some general comments on possible extensions.

Section 4 introduces an evolutionary dynamics for symmetric games. Section 4.1 describes the replicator dynamics and its relation to evolutionary games. Some important technical issues are also summarized. Section 4.2 establishes the relation between the replicator dynamics and a normal-form game using the concepts of Nash equilibria and *strongly uninvadable strategies*. Section 4.3 presents a brief review of stability results for the replicator dynamics. Section 4.4 establishes an important relationship between Nash equilibria and the critical points of the replicator dynamics. Section 4.5 proposes examples to illustrate our results. Finally, we conclude in Section 4.6 with some general comments on possible extensions of our results.

978-1-009-47230-2 — Evolutionary Games and the Replicator Dynamics
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Section 5 proposes approximation schemes for the replicator dynamics in measure spaces, including the approximation by dynamical systems in finite-dimensional spaces (see Section 5.1). Section 5.2 presents an approximation theorem in the strong form, using the variational norm; and Section 5.3 presents an approximation theorem in the weak form, using the Kantorovich–Rubinstein norm. Section 5.4 illustrates with some examples our results. Finally, we conclude in Section 5.5 with some comments on other possible approximations.

Section 6 studies the replicator dynamics as a limit of a sequence of Markov processes. Section 6.1 presents notation and technical requirements. Section 6.2 shows a technique proposed by Kolokoltsov (2006, 2010) to approximate a sequence of pure jump models of binary interaction (in a Banach space), via a deterministic dynamical system. Section 6.3 uses techniques of Section 6.2 to establish conditions under which the replicator dynamics are the limit of a sequence of Markov processes.

Section 7 presents a summary of contributions and future perspectives. Finally, the online appendix (available at www.cambridge.org/evolutionary-games) contains facts on metrics on spaces of probability measures, and the proof of some technical results.

1.3 Technical Preliminaries

1.3.1 Spaces of Signed Measures

Consider a separable metric space A and its Borel σ-algebra $\mathcal{B}(A)$. Let $\mathbb{M}(A)$ be the Banach space of finite signed measures μ on $\mathcal{B}(A)$ endowed with the total variation norm

$$\|\mu\| := \sup_{\|f\|\leq 1} \left| \int_A f(a)\mu(da) \right| = |\mu|(A), \tag{1}$$

where $|\mu| = \mu^+ + \mu^-$ denotes the total variation of μ, and μ^+, μ^- stand for the positive and negative parts of μ, respectively. The supremum in (1) is taken over functions in the Banach space $\mathbb{B}(A)$ of real-valued bounded measurable functions on A, endowed with the supremum norm

$$\|f\| := \sup_{a\in A} |f(a)|. \tag{2}$$

Consider the subset $\mathbb{C}_B(A) \subset \mathbb{B}(A)$ of all real-valued continuous and bounded functions on A, and the dual pair $(\mathbb{C}_B(A), \mathbb{M}(A))$ given by the bilinear form $\langle \cdot, \cdot \rangle : \mathbb{C}_B(A) \times \mathbb{M}(A) \to \mathbb{R}$

$$\langle g, \mu \rangle := \int_A g(a)\mu(da). \tag{3}$$

We consider the *weak topology* on $\mathbb{M}(A)$ (induced by $\mathbb{C}_B(A)$), that is, the topology under which the elements of $\mathbb{C}_B(A)$, when regarded as linear functionals

Cambridge University Press & Assessment
978-1-009-47230-2 — Evolutionary Games and the Replicator Dynamics
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$\langle g, \cdot \rangle$ on $\mathbb{M}(A)$, are continuous. In this topology a neighborhood of a point $\mu \in \mathbb{M}(A)$ is of the form

$$\mathcal{V}_{\epsilon}^{\mathcal{H}}(\mu) := \left\{ \nu \in \mathbb{M}(A): \ |\langle g, \nu - \mu \rangle| < \epsilon \ \ \forall g \in \mathcal{H} \right\} \tag{4}$$

for $\epsilon > 0$ and $\mathcal{H}$ a finite subset of $\mathbb{C}_B(A)$.

Definition 1 *A sequence of measures $\mu_n \in \mathbb{M}(A)$ is said to be* weakly convergent *if there exists $\mu \in \mathbb{M}(A)$ such that*

$$\lim_{n\to\infty} \int_A g(a)\mu_n(da) = \int_X g(a)\mu(da) \tag{5}$$

for all g in $\mathbb{C}_B(A)$. If $\mathbb{M}(A)$ is replaced by the space $\mathbb{P}(A)$ of probability measures on A, sometimes we say that μ_n converges in distribution to μ.

1.3.2 Metrics on $\mathbb{P}(A)$

There are many metrics that metrize the weak topology. The following metrics are particularly useful. (For details see, for instance, Shiryaev (1996), Billingsley (2013), or Villani (2008)). This subsection will be used in Section 4; the reader can skip it and come back to it later.

Let A be a separable metric space with a metric ϑ, and $\mathbb{P}(A)$ the set of probability measures on A. For any $\mu, \nu \in \mathbb{P}(A)$ we define the following metrics on $\mathbb{P}(A)$.

(i) **The Prokhorov metric** r_p, defined as

$$r_p(\mu, \nu) := \inf\{\alpha > 0: \ \mu(E) \le \nu(E_\alpha) + \alpha \ \text{ and } \ \nu(E) \le \mu(E_\alpha) + \alpha\}, \tag{6}$$

where, for $\alpha > 0$, $E_\alpha := \{a \in A: \vartheta(a, E) < \alpha\}$ if $E \neq \phi$. Here ϕ is the empty set, and

$$\vartheta(a, E) := \inf\{\vartheta(a, a'): \ a' \in E\}.$$

(ii) **The bounded Lipschitz metric** r_{bl}, defined as

$$r_{bl}(\mu, \nu) := \sup_{f \in \mathbb{L}_B(A)} \left\{ \int_A f(a)\mu(da) - \int_A f(a)\nu(da): \ \|f\|_{BL} \le 1 \right\}, \tag{7}$$

where $(\mathbb{L}_B(A), \|\cdot\|_{BL})$ is the space of bounded, continuous, and real-valued functions on A that satisfy the Lipschitz condition

$$\|f\|_L := \sup \frac{|f(a) - f(b)|}{\vartheta(a, b)} < \infty, \tag{8}$$

where the supremum is over all $a \neq b$. For any $f \in \mathbb{L}_B(A)$, the norm $\|f\|_{BL}$ is defined as

$$\|f\|_{BL} := \|f\| + \|f\|_L. \tag{9}$$

(*iii*) **The Kantorovich–Rubinstein metric** r_{kr}, defined as

$$r_{kr}(\mu,\nu) := \sup_{f\in\mathbb{L}(A)} \left\{ \int_A f(a)\mu(da) - \int_A f(a)\nu(da) : \|f\|_L \le 1 \right\}, \tag{10}$$

where $(\mathbb{L}(A), \|\cdot\|_L)$ is the space of continuous real-valued functions on A that satisfy the Lipschitz condition (8). Let a_0 be a fixed point in A, and

$$\mathbb{M}_K(A) := \left\{ \mu \in \mathbb{M}(A) : \sup_{f\in\mathbb{L}(A)} \int_A |f(a)|\mu(da) < \infty \right\}.$$

Then the Kantorovich–Rubinstein metric r_{kr} can be extended as a norm on $\mathbb{M}_K(A)$ defined as

$$\|\mu\|_{kr} := |\mu(A)| + \sup_{f\in\mathbb{L}(A)} \left\{ \int_A f(a)\mu(da) : \|f\|_L \le 1,\ f(a_0) = 0 \right\}, \tag{11}$$

for any μ in $\mathbb{M}_K(A)$ (see Bogachev (2007), chapter 8). Note that for any $\mu,\nu \in \mathbb{P}(A)$ $r_{kr}(\mu,\nu) = \|\mu-\nu\|_{kr}$, since

$$\begin{aligned}
&\sup_{f\in\mathbb{L}(A)} \left\{ \int_A f(a)\mu(da) - \int_A f(a)\nu(da) : \|f\|_L \le 1 \right\} \\
&= \sup_{f\in\mathbb{L}(A)} \left\{ \int_A \Big(f(a) - f(a_0)\Big)\mu(da) - \int_A \Big(f(a) - f(a_0)\Big)\nu(da) : \|f\|_L \le 1 \right\} \\
&= \sup_{g\in\mathbb{L}(A)} \left\{ \int_A g(a)\mu(da) - \int_A g(a)\nu(da) : \|g\|_L \le 1,\ g(a_0) = 0 \right\}.
\end{aligned}$$

(*iv*) Let us suppose that the separable metric space A is also complete (that is, a so-called Polish space), and let a_0 be a fixed point in A. For each p with $1 \le p < \infty$, we define the space $\mathbb{P}_p(A)$ as

$$\mathbb{P}_p(A) := \left\{ \mu \in \mathbb{P}(A) : \int_A [\vartheta(a,a_0)]^p \mu(da) < \infty \right\}.$$

The L^p**-Wasserstein distance** r_{w_p} between μ and ν in $\mathbb{P}_p(A)$ is defined by

$$r_{w_p}(\mu,\nu) := \left[\inf_{\pi\in\Pi} \int_A \int_A [\vartheta(a,b)]^p \pi(da,db) \right]^{\frac{1}{p}}, \tag{12}$$

where Π is the set of probability measures on $A \times A$ with marginals μ and ν. In particular, when $p = 1$, we write the L^1**-Wasserstein distance** r_{w_1} as r_w and in addition we have that $r_w = r_{kr}$ on $\mathbb{P}(A)$.

Remark 2 *In the rest of this work we will denote by* r_{w^*} **any metric that metrizes the weak topology** *on* $\mathbb{P}(A)$ *(not to be confused with the notation* r_w *of the* L^1*-Wasserstein distance). Moreover, we denote by* r **any metric on** $\mathbb{P}(A)$ *that is either the total variation norm* (1) *or any distance that metrizes the weak topology. An open ball in the metric space* $(\mathbb{P}(A), r)$ *is defined in the classical form*

$$\mathcal{V}^r_\alpha(\mu) := \Big\{\nu \in \mathbb{P}(A) \colon r(\nu, \mu) < \alpha\Big\}, \tag{13}$$

where $\alpha > 0$.

Remark 3 *Let* A *be a separable metric space, and* r_{w^*} *any distance that metrizes the weak topology* τ_{w^*} *in* $\mathbb{P}(A)$. *Let* μ *be any measure in* $\mathbb{P}(A)$, *and consider the family* $\mathcal{V}^{\mathcal{H}}(\mu)$ *of neighborhoods* $\mathcal{V}^{\mathcal{H}}_\epsilon(\mu)$ *of the form* (4). *In addition, consider the family* $\mathcal{V}^{r_{w^*}}(\mu)$ *of the open balls* $\mathcal{V}^{r_{w^*}}_\alpha(\mu)$ *of the form* (13). *Both families* $\mathcal{V}^{\mathcal{H}}(\mu)$ *and* $\mathcal{V}^{r_{w^*}}(\mu)$ *are* neighborhood basis *for* μ *in the space* $(\mathbb{P}(A), \tau_{w^*})$. *For details see Pedersen (2012), chapters I–II.*

Moreover, a neighborhood $\mathcal{V}^{\mathcal{H}}_\epsilon(\mu)$ *for* μ *is contained in some* open ball $\mathcal{V}^{r_{w^*}}_\alpha(\mu)$ with center μ. *The inverse is also true, that is, any open ball* $\mathcal{V}^{r_{w^*}}_\alpha(\mu)$ *is contained in some neighborhood* $\mathcal{V}^{\mathcal{H}}_\epsilon(\mu)$.

1.3.3 Product Spaces

Consider two separable metric spaces X and Y with their Borel σ-algebras $\mathcal{B}(X)$ and $\mathcal{B}(Y)$. We denote by $\sigma[X \times Y]$ the σ-algebra on $X \times Y$ generated by the Cartesian product $\mathcal{B}(X) \times \mathcal{B}(Y)$. For $\mu \in \mathbb{M}(X)$ and $\nu \in \mathbb{M}(Y)$, we denote their product by $\mu \times \nu \in \mathbb{M}(X \times Y)$.

Proposition 4 *For* $\mu \in \mathbb{M}(X)$ *and* $\nu \in \mathbb{M}(Y)$, *it holds that*

$$\|\mu \times \nu\| \leq \|\mu\|\|\nu\|. \tag{14}$$

As a consequence, $\mu \times \nu$ *is in* $\mathbb{M}(X \times Y)$.

Proof See Heidergott and Leahu (2010), lemma 4.2. □

Now consider a finite family of metric spaces $\{X_i\}_{i=1}^n$ and their σ-algebras $\mathcal{B}(X_i)$, as well as the Banach spaces $(\mathbb{M}(X_i), \|\cdot\|)$ and $(\mathbb{M}_K(X_i), \|\cdot\|_{kr})$. For $i = 1, \ldots, n$, let $\mu_i \in \mathbb{M}(X_i)$. Consider the elements $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ in the product space $\mathbb{M}(X_1) \times \mathbb{M}(X_2) \times \cdots \times \mathbb{M}(X_n)$ for which

$$\|\mu\|_\infty = \|(\mu_1, \ldots, \mu_n)\|_\infty := \max_{1 \leq i \leq n} \|\mu_i\| < \infty. \tag{15}$$

These elements form a Banach space with $\|\cdot\|_\infty$ as a norm. We call it the *direct product* of the Banach spaces $\mathbb{M}(X_i)$. We can similarly define the Banach space $(\mathbb{M}_K(X_1)\times\cdots\times\mathbb{M}_K(X_n), \|\cdot\|_\infty^{kr})$, where

$$\|\mu\|_\infty^{kr} := \max_{1\le i\le n} \|\mu_i\|_{kr} < \infty. \tag{16}$$

1.3.4 Differentiability

Definition 5 *Let A be a separable metric space. We say that a mapping $\mu\colon [0,\infty)\to\mathbb{M}(A)$ is* strongly differentiable *if there exists $\mu'(t)\in\mathbb{M}(A)$ such that, for every $t>0$,*

$$\lim_{\epsilon\to 0}\left\|\frac{\mu(t+\epsilon)-\mu(t)}{\epsilon}-\mu'(t)\right\| = 0. \tag{17}$$

Note that, by (1)*, the left-hand side in* (17) *can be expressed as*

$$\lim_{\epsilon\to 0}\sup_{\|g\|\le 1}\left|\frac{1}{\epsilon}\left[\int_A g(a)\mu(t+\epsilon,da)-\int_A g(a)\mu(t,da)\right]-\int_A g(a)\mu'(t,da)\right|.$$

The signed measure μ' in (17) *is called* strong derivative.

For weak differentiability, see Remark 47.

1.4 Comments

This section presented a general introduction and summary of this Element. In addition, some technical preliminaries to be used in the following sections were presented. The only remaining information to be included is some references addressing evolutionary games in an explicit and comprehensive manner. First, we mention references for evolutionary games with *finite* strategy spaces. Webb (2007) and Weibull (1997) are two good introductory books; Hofbauer and Sigmund (1998) and Sandholm (2010) are two books that address a larger number of topics on evolutionary games; Cressman (2003) uses techniques based on subgame decompositions of *extensive form games* to analyze convergence results for evolutionary dynamics.

Regarding books dealing with evolutionary games with measurable strategy spaces we can only mention Bomze and Pötscher (1989). Nevertheless, this book was written in 1989 and so it does not address several subsequent results that have been developed in this theory. Other references on theoretical advances of this topic are mentioned in the introduction. Most of them, however, only touch theoretical aspects, and there are few bibliographies about applications; for example, oligopoly theory, Rabanal (2017), public goods models, Rabanal (2017), and preferences economic theory, Heifetz et al. (2007) and Norman (2012).