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Groups and Graphs, Designs and Dynamics

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Preface

The conference and PhD–Masters summer school G2D2 (Groups and Graphs, Designs and Dynamics), part of the G2 series of conferences[1], was held in the Three Gorges Mathematical Research Center in Yichang, China, in 2019.

The material presented in the conference provided participants with a Chinese banquet of beautiful and important mathematics. The main courses were short lecture courses on the four topics in the title of the conference, records of which make up the present volume. In addition there were introductory lectures on each of the four topics, together with many invited and contributed talks. We would like to invite you warmly to participate in this banquet.

The base of a Chinese banquet is rice, and this role is played here by linear algebra. Dating from the nineteenth century, linear algebra is now a central part of mathematics and is fundamental to all the areas described here; indeed, it forms a unifying thread running through four chapters describing very different parts of mathematics.

The basic interpretation of a square matrix in linear algebra is that it represents a linear transformation from a vector space to itself; we are concerned with properties of the transformation independent of the means of representation. A natural extension of this is to consider a set of such matrices, and there are good reasons for taking the set to be a group; this leads us to the important topic of group representation. Its importance is indicated by comparing the two editions of William Burnside's book on finite groups, the first such book in English. In the first edition, he explains why he has omitted all mention of representation theory, since "it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations." However, by the second edition, he explains how recent re-

[1] The history of this series of conferences, designed to encourage international cooperation, can be found at `https://ekonsta.github.io/Slides/G2-series.pdf`

sults (by Frobenius, Schur and Burnside himself) have led him to change his opinion, and devote more than half the book to representation theory.

The chapter by Tullio Ceccherini-Silberstein, Fabio Scarabotti and Filippo Tolli is an exposition of representation theory, strongly influenced by its applications to harmonic analysis, random walks, Gelfand pairs, and representations of general linear groups.

In general, the concept of positivity links algebra and analysis. Equipped with a $*$-operation (involution), an algebra allows us to define a positive linear functional, which plays the role of expectation (mean value) in probability theory. In this aspect, the idea of algebraic probability emerges and provides a quantum probabilistic approach to the study of discrete structures. The adjacency matrix of a simple graph is a real symmetric nonnegative matrix, so its spectrum is an important isomorphism invariant, which carries important information about the graph. The chapter by Nobuaki Obata shows how the spectra of graphs in various families which have a limit object can be calculated, using the techniques of quantum probability theory.

The purpose of statistics is to derive useful information from data. If the data come from an experiment, the scientist has the option of using a design that will more efficiently yield the information, because the random variables representing estimated values of the parameters have smallest possible variance. Since this is a multidimensional optimization problem, there is no unique solution, and several optimality measures have been proposed. The chapter by R. A. Bailey and Peter Cameron shows that, in the case where a block design is used, all common optimality criteria can be expressed in terms of the spectrum of the Laplacian matrix of a certain graph, the concurrence graph of the design. The Laplacian has applications in many other fields such as electrical networks and random walks, and these applications are also described.

Symbolic dynamics grew from the work of Poincaré in the early twentieth century; for some questions, complicated continuous dynamical systems can be replaced by subsystems of the shift operation on the set of infinite sequences over a finite alphabet. In this case, unlike the others, the linear algebra required is not “off-the-shelf”, since it involves matrices over a semiring such as the natural numbers. This leads naturally to the concepts of stability and K-theory, all of which are explained in detail in the chapter by Mike Boyle and Scott Schmieding, after an introduction to topological dynamics.

The authors have all been aware of connections between their topics and other parts of mathematics, whether these be representation theory and Markov chains, graph spectra and quantum probability, optimal designs and electrical circuits, or symbolic dynamics and algebraic K-theory.

We thank our authors for their work on the project, and we hope that bringing these surveys on different topics into a single book will help highlight the similarities between them as well as their individual aesthetic attraction. On this, let us quote four mathematicians:

I am just a student all my life. From the very beginning of my life I was trying to learn. And for example now, when listening to the talks and reading notes of this conference, I discover how much I still do not know and have to learn. Therefore, I am always learning. In this sense I am a student. Never a "Führer".

Israel M. Gelfand, a dinner talk in the conference "The Unity of Mathematics"

The cross-fertilization of ideas is crucial for the health of the science and mathematics.

Mikhael Gromov, Possible Trends in Mathematics in the Coming Decades, an open letter addressed to the French mathematical community in 1998

Mathematics has three purposes. It delivers an instrument for the study of nature, but this is not all. It has a philosophical purpose and, I dare say, an aesthetic purpose. These purposes can not be separated and the best way to achieve one purpose is to aim at the other ones, or at least not to lose sight of them.

Henri Poincaré

My own mathematical works are always quite unsystematic, without mode or connection. Expression and shape are almost more to me than knowledge itself. But I believe that, leaving aside my own peculiar nature, there is in mathematics itself, in contrast to the experimental disciplines, a character which is nearer to that of free creative art.

Hermann Weyl

We hope that our readers will experience some of what these mathematicians describe.

The chapters of the book are in the order of the subjects in the title: Groups, Graphs, Designs, Dynamics. Although all have their roots in linear algebra, and there are a number of cross-references, we stress that the chapters are independent, and can be read in any order. Also, many topics, ranging from major fields like probability theory to niche subjects like distance-regular graphs, recur in several chapters. We invite readers who are expert in one of our four topics to explore some of the others.

According to Donald Knuth in *The Art of Computer Programming*,

The book *Dynamic Programming* by Richard Bellman is an important, pioneering work in which a group of problems is collected together at the end of some chapters under the heading "Exercises and Research Problems," with extremely trivial questions appearing in the midst of deep, unsolved problems. It is rumored that someone once asked

Dr. Bellman how to tell the exercises apart from the research problems, and he replied: "If you can solve it, it is an exercise; otherwise it's a research problem."

We have not inflicted this on our readers: exercises and problems are clearly labelled. But some authors have scattered exercises through the text, while others have gathered them at the end of sections. In particular, in the first chapter, the exercises form an integral part of the development of the theory. In general, we urge readers to try the exercises for themselves, as students at the original courses were encouraged to do.

We want to express our gratitude to many people who have made this book possible, especially the authors for the care and attention with which they have produced their chapters. We thank Xiaoli Wu at Higher Education Press and David Liu and Anna Scriven at Cambridge University Press for help in turning four individual chapters into a coherent book. For the G2D2 event, we thank Xuemei Deng, Zongzhu Lin, Shuang Yi from Three Gorges Mathematical Research Center and Sergey Goryainov, Da Zhao, Yinfeng Zhu from Shanghai Jiao Tong University; we also acknowledge financial support from Three Gorges Mathematical Research Center, National Natural Science Foundation of China and International Linear Algebra Society.

R. A. Bailey, Peter J. Cameron, Yaokun Wu