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Topics in representation theory of finite groups

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Abstract

This is an introduction to representation theory and harmonic analysis on finite groups. This includes, in particular, Gelfand pairs (with applications to diffusion processes *à la* Diaconis) and induced representations (focusing on the little group method of Mackey and Wigner). We also discuss Laplace operators and spectral theory of finite regular graphs. In the last part, we present the representation theory of $\mathrm{GL}(2,\mathbb{F}_q)$, the general linear group of invertible 2×2 matrices with coefficients in a finite field with q elements. More precisely, we revisit the classical Gelfand–Graev representation of $\mathrm{GL}(2,\mathbb{F}_q)$ in terms of the so-called multiplicity-free triples and their associated Hecke algebras. The presentation is not fully self-contained: most of the basic and elementary facts are proved in detail, some others are left as exercises, while, for more advanced results with no proof, precise references are provided.

Keywords: finite group, group representation, character, Gelfand pair, spherical function, spherical Fourier transform, Mackey–Wigner little group method, Markov chain, random walk, Ehrenfest diffusion process, ergodic theorem, finite graph, spectral graph theory, Laplace operator, distance-regular graph, strongly regular graph, association scheme, finite field, affine group over a finite field, general linear group over a finite field, Gelfand–Graev representation, multplicity-free triple, Hecke algebra

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1.1 Introduction

The present text constitutes an expanded and more detailed exposition of the lecture notes of a course on Representation Theory delivered by the first named author at the International Conference and PhD-Master Summer School on Groups and Graphs, Designs and Dynamics (G2D2) held in Yichang (China) in August 2019.

One of the main features of Harmonic Analysis is the study of linear operators that are invariant with respect to the action of a group. In the classical abelian setting, for instance, this is used to express the solutions of a constant coefficients differential equation (such as the heat equation) in terms of infinite sums of exponentials (Fourier series).

Here, we consider a finite (possibly non-abelian) counterpart. Let G be a finite group, let $K \leq G$ be a subgroup, and consider the G-module $L(G/K)$ of all complex valued functions on the (finite) homogenous space G/K of left cosets of K in G. The corresponding space of linear G-invariant operators we alluded to above, the so-called commutant $\mathrm{End}_G(L(G/K))$, bears a natural structure of an involutive unital algebra that turns out to be isomorphic to the algebra ${}^K L(G)^K$ of all bi-K-invariant complex valued functions on G. When these algebras are commutative, we say that (G,K) is a Gelfand pair: the terminology originates from the seminal paper by I. M. Gelfand [40] in the setting of Lie groups. Finite Gelfand pairs, when G is a Weyl group or a Chevalley group over a finite field, or the symmetric group $S_n = \mathrm{Sym}(\{1,2,\ldots,n\})$, were studied by Ph. Delsarte [25], motivated by applications to association schemes of coding theory, Ch F. Dunkl [30, 31, 32, 33] and D. Stanton [67] with relevant contributions to the theory of special functions, E. Bannai and T. Ito [3] who initiated Algebraic Combinatorics, J. Saxl [59] in the study of Finite Geometries and Designs, and A. Terras [69] with applications to number theory. A special mention deserves the work in Probability Theory by P. Diaconis and collaborators [26] with remarkable applications to the study of diffusion processes and asymptotic behaviour of finite Markov chains. A. Okounkov and A. M. Vershik [55] (see also [16]) used methods from the theory of finite Gelfand pairs in order to give a new approach to the representation theory of the symmetric groups. Further expositions of the theory of finite Gelfand pairs and association schemes can be found in the monographs by R. A. Bailey [2], P.-H. Zieschang [73], as well as in the survey paper [14] and in our first monograph [15]. We conclude this bibliographical overview by mentioning the work of R. I. Grigorchuk [43] (see also [5, 23, 24]) in connection with the theory of the so-called self-similar groups.

Given a Gelfand pair (G,K), the simultaneous diagonalization of all G-

invariant operators can be achieved by means of a particular basis of ${}^{K}L(G)^{K}$. The elements of this basis, called spherical functions, are the analogues of the exponentials in the classical case and can be defined both intrinsically and as matrix coefficients of particular representations (the spherical representations). Besides the trivial though interesting case when the group G is abelian, an important example of a Gelfand pair is given by $(G \times G, \widetilde{G})$, with $\widetilde{G}$ the diagonal subgroup: in this case, the spherical functions are nothing but the normalized characters of G, showing that the theory of central functions on a group can be treated in the setting of the Gelfand pairs, as a particular case.

By virtue of the Ergodic Theorem, the rate of convergence to the stationary distribution of the n-step distributions μ_n of a finite (ergodic and symmetric) Markov chain can be estimated in terms of the second largest eigenvalue modulus of the corresponding transition matrix. An example of a Gelfand pair is $(S_n, S_k \times S_{n-k})$, where $S_n = \mathrm{Sym}(\{1,2,\ldots,n\})$ is the symmetric group of degree n, and, for $1 \leq k \leq n/2$, we regard $S_k = \mathrm{Sym}(\{1,2,\ldots,k\})$ and $S_{n-k} = \mathrm{Sym}(\{k+1,k+2,\ldots,n\})$ as subgroups of S_n. In the 80s Diaconis and Shahshahani [28] (see also [14, 15]), were able to use this Gelfand pair to find very precise asymptotics of $(\mu_n)_{n\in\mathbb{N}}$ for the Bernoulli–Laplace model of diffusion. In particular, they showed that an interesting phenomenon occurs: the transition from order to chaos is concentrated in a relatively small interval of time: this is the *cut-off phenomenon*. Other important examples, where the theory of spherical functions plays a central role, are the Ehrenfest model of diffusion (see Section 1.5.2) and the random transpositions model [26, 27, 14, 15].

The G-module $L(G/K)$ can be seen as the representation space of the induced representation $\mathrm{Ind}_K^G \iota_K$ of the trivial representation ι_K of K, and we have that (G,K) is a Gelfand pair if and only if $\mathrm{Ind}_K^G \iota_K$ decomposes without multiplicity. More generally, if θ is an irreducible K-representation, the algebra $\mathrm{End}_G(\mathrm{Ind}_K^G \theta)$ of intertwiners is isomorphic to a suitable convolution algebra $\mathscr{H}(G,K,\theta)$ of complex valued functions on G, and we say that (G,K,θ) is a multiplicity-free triple if these algebras are commutative; equivalently, if $\mathrm{Ind}_K^G \theta$ decomposes without multiplicity. Multiplicity-free triples were partially studied by I. G. Macdonald [50], by D. Bump and D. Ginzburg [9], and in [19, Chapter 13] when $\dim\theta = 1$; a generalization to higher dimensions, with a complete analysis of the spherical functions, is treated in our papers [61, 62, 63, 64] and the recent monograph [20]. An earlier application, where a problem of Diaconis on the Bernoulli-Laplace diffusion model with many urns was solved, was presented in the second named author's PhD thesis and published in [60]. As pointed out in [19, Chapter 14], our theory of multiplicity-free triples shed light on the representation theory of $\mathrm{GL}(2,\mathbb{F}_q)$, the general linear group of 2×2 matrices with coefficients in the field with q elements, as developed by I. I. Piatetski-Shapiro in [57].

These lecture notes are organized as follows. In Section 1.2, we briefly recall the basics of the representation theory of finite groups: this includes Schur's lemma, some character theory, and the Peter–Weyl theorem. In Sections 1.2.2, 1.2.3, and 1.2.4 we study Gelfand pairs in detail, focusing on spherical functions, the spherical Fourier transform, and the harmonic analysis of invariant operators. Then, in Sections 1.5.1 and 1.5.2 we present the applications of Gelfand pairs to Markov chains, culminating in the celebrated Diaconis–Shahshahani upper-bound lemma, and describe the asymptotics for the Ehrenfest model of diffusion. In Sections 1.6.1 and 1.6.2 we study induced representations, Frobenius reciprocity, and Mackey theory, and then, in Section 1.6.3, we apply this machinery to obtain the Mackey–Wigner little group method. In Section 1.6.4 we introduce the Hecke algebras $\widetilde{\mathscr{H}}(G,K,\theta)$ and $\mathscr{H}(G,K,\theta)$ and show that they are both isomorphic to the commutant $\mathrm{End}_G(\mathrm{Ind}_K^G \theta)$. In Section 1.6.5 we then define multiplicity-free triples and present their general theory. After a short overview of the basics of finite fields and their characters (Section 1.7.1), as an application of the little group method of Mackey and Wigner we describe all irreducible representations of $\mathrm{Aff}(\mathbb{F}_q)$, the affine group over the field with q elements. The last two sections are devoted to the general linear group $\mathrm{GL}(2,\mathbb{F}_q)$ and its representations: in relation with the latter, we limit ourselves to the description of the decomposition of the Gelfand–Graev representation.

Our presentation is mostly self-contained. However, for the sake of brevity, some of the proofs are either omitted (but with clear references for a complete exposition), or sketched, or left as an exercise to the reader. Several other exercises are proposed as complements and further developments.

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1.2 Representation theory and harmonic analysis on finite groups

In this section, we present the basics of the representation theory of finite groups and we introduce and study the notion of a finite Gelfand pair, thus providing a setting for a suitable extension of the classical Fourier analysis.

Our exposition is inspired by Diaconis' book [26] and to Figà–Talamanca's lecture notes [37] and our monographs [15, 19]. We also took a particular benefit from the monographs by Alperin and Bell [1], Fulton and Harris [39], Isaacs [46], Naimark and Stern [52], Serre [65], Simon [66], and Sternberg [68]. Expositions of the theory of Gelfand pairs are also presented in the monographs by J. Dieudonné [29], H. Dym and H. P. McKean [34] , J. Faraut [36], A. Figà-Talamanca and C. Nebbia [38], S. Helgason [44] and J. Wolf [71] for the general case of locally compact groups.

1.2.1 Representations

Let G be a finite group.

Definition 1.2.1 (Representation) A *representation* of G (also called a G-*representation*) is a pair (ρ, V), where V is a finite dimensional complex vector space and $\rho\colon G \to \mathrm{GL}(V)$ is a group homomorphism from G into the group $\mathrm{GL}(V)$ of all *invertible linear transformations* of V.

If (ρ, V) is a representation of G, then one has:

- $\rho(1_G) = I_V$
- $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$
- $\rho(g^{-1}) = \rho(g)^{-1}$
- $\rho(g)(av + bw) = a\rho(g)v + b\rho(g)w$

for all $g, g_1, g_2 \in G$, $v, w \in V$, and $a, b \in \mathbb{C}$, where $1_G \in G$ is the identity element and $I_V\colon V \to V$ is the identity transformation.

Equivalently, a representation can be viewed as an *action* $\alpha\colon G \times V \to V$ of G on V by linear transformations by setting $\alpha(g, v) := \rho(g)v$ for all $g \in G$ and $v \in V$.

In the following, for the sake of brevity, when a given representation (ρ, V) is clear from the context, we shall denote it simply by either ρ or V.

The dimension $d_\rho := \dim V$ of the vector space V is called the *dimension* of ρ.

Definition 1.2.2 (Sub-representation) Let (ρ, V) be a G-representation. A subspace $W \leq V$ is G-*invariant* if $\rho(g)w \in W$ for all $g \in G$ and $w \in W$. The pair (ρ_W, W), where $\rho_W(g) := \rho(g)|_W$ for all $g \in G$, is a G-representation, called a *sub-representation* of (ρ, V). We shall then write $(\rho_W, W) \leq (\rho, V)$.

Clearly, $d_{\rho_W} \leq d_\rho$.

Definition 1.2.3 (Irreducible representation) A G-representation (ρ, V) is *irreducible* if V admits no nontrivial G-invariant subspaces, that is, the only G-invariant subspaces $W \leq V$ are $W = \{0\}$ and $W = V$.

We denote by $\mathrm{Irr}(G)$ the set of all irreducible representations of G.

The representation of dimension zero is considered to be neither reducible nor irreducible, just as the number 1 is considered to be neither composite nor prime.

It is obvious that every one-dimensional representation is irreducible.

Definition 1.2.4 (Equivalent representations) Two G-representations (ρ_1, V_1) and (ρ_2, V_2) are *equivalent* if there exists a linear isomorphism $T\colon V_1 \to V_2$ such that

$$T \circ \rho_1(g) = \rho_2(g) \circ T$$

for all $g \in G$. We then write $\rho_1 \sim \rho_2$. We shall refer to T as to an *intertwining isomorphism*.

If (ρ_1, V_1) is equivalent to a sub-representation of (ρ_2, V_2) we write $\rho_1 \preceq \rho_2$, and we say that ρ_1 is *contained* in ρ_2.

Note that $\sim$ is an equivalence relation in the set of all G-representations, which preserves irreducibility and dimension (exercise).

Definition 1.2.5 (Unitary representation) Suppose that a complex vector space V is equipped with an inner product $\langle \cdot, \cdot \rangle_V$. A G-representation (ρ, V) is *unitary* if, for every $g \in G$, the linear operator $\rho(g)$ is unitary, that is,

$$\langle \rho(g)v_1, \rho(g)v_2 \rangle_V = \langle v_1, v_2 \rangle_V$$

for all $v_1, v_2 \in V$.

Note that, if (ρ, V) is a unitary representation, then

- $\rho(g^{-1}) = \rho(g)^*$

for all $g \in G$, where * denotes the *adjoint* operation.

Exercise 1.2.6 (Unitarizability of representations) Suppose that a complex vector space V is equipped with an inner product $\langle \cdot, \cdot \rangle_V$. Let (ρ, V) be a G-representation. Then when equipping V with the new inner product $(\cdot, \cdot)_V$ defined by

$$(v_1, v_2)_V := \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)v_1, \rho(g)v_2 \rangle_V$$

for all $v_1, v_2 \in V$, the representation (ρ, V) becomes unitary.

By virtue of the previous exercise, from now on, we shall consider only unitary representations. This will not affect equivalence as the next exercise shows.

Exercise 1.2.7 Let (ρ_1, V_1) and (ρ_2, V_2) be two unitary G-representations. Suppose that $\rho_1 \sim \rho_2$. Then there exists a unitary operator $U \colon V_1 \to V_2$ such that

$$U \circ \rho_1(g) = \rho_2(g) \circ U$$

for all $g \in G$.

Hint: Use the *polar decomposition* $T = U|T|$ for an intertwining isomorphism $T \colon V_1 \to V_2$ (for more details, see [19, Lemma 10.1.4]).

We can rephrase the result in the above exercise by saying that two equivalent unitary representations are *unitarily equivalent*.

Definition 1.2.8 (Dual of a group) The *dual* of the group G is the quotient $\widehat{G} := \mathrm{Irr}(G)/\sim$. In the following we shall also refer to $\widehat{G}$ as to a complete set of irreducible pairwise non-equivalent G-representations.

We shall see later (cf. Theorem 1.2.36) that $|\widehat{G}| < \infty$.

Definition 1.2.9 (Direct sum) Let (ρ_1, V_1) and (ρ_2, V_2) be two G-representations. We equip $V := V_1 \oplus V_2$ with the inner product $\langle \cdot, \cdot \rangle_V$ defined by setting

$$\langle v_1 + v_2, v_1' + v_2' \rangle_V := \langle v_1, v_1' \rangle_{V_1} + \langle v_2, v_2' \rangle_{V_2}$$

for all $v_1, v_1' \in V_1$ and $v_2, v_2' \in V_2$. The (unitary) G-representation (ρ, V) defined by setting

$$\rho(g)(v_1 + v_2) := \rho_1(g)v_1 + \rho_2(g)v_2$$

for all $g \in G$ and $v_1 \in V_1$, $v_2 \in V_2$, is called the *direct sum* of (ρ_1, V_1) and (ρ_2, V_2) and is denoted by $(\rho_1 \oplus \rho_2, V_1 \oplus V_2)$.

Note that $d_{\rho_1 \oplus \rho_2} = d_{\rho_1} + d_{\rho_2}$ and that $\rho_i \preceq \rho_1 \oplus \rho_2$ for $i = 1, 2$.

Definition 1.2.10 (Conjugate representation) Let (ρ, V) be a G-representation and let V' denote the dual vector space. The *conjugate representation* of ρ is the unitary representation (ρ', V') defined by setting

$$[\rho'(g)f](v) := f(\rho(g^{-1})v)$$

for all $g \in G$, $f \in V'$, and $v \in V$.

It is an exercise to check that ρ' is unitary (resp. irreducible) if and only if ρ is unitary (resp. irreducible).

Exercise 1.2.11 (Orthogonal complement) Suppose that (ρ,V) is a (unitary) G-representation and let $W \leq V$ be a nontrivial G-invariant subspace. Show that

$$W^{\perp} = \{v \in V : \langle v,w\rangle_V = 0 \text{ for all } w \in W\}$$

is also G-invariant. Deduce that $\rho = \rho_W \oplus \rho_{W^\perp}$.

From the above exercise and an obvious inductive argument, one immediately deduces the following:

Theorem 1.2.12 *Every G-representation is the direct sum of finitely many irreducible G-representations.* □

The above theorem may be rephrased as follows. Suppose that (ρ,V) is a G-representation. Then there exist a positive integer n and (not necessarily distinct) $\rho_1,\rho_2,\dots,\rho_n \in \widehat{G}$ such that $\rho \sim \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_n$.

Example 1.2.13 (Trivial representation) The *trivial representation* of a group G, denoted $(\iota_G,\mathbb{C})$, is the one-dimensional representation defined by setting $\iota_G(g) = \mathrm{Id}_{\mathbb{C}}$ for all $g \in G$.

Given a finite group G, we denote by $L(G)$ the complex vector space of all functions $f\colon G \to \mathbb{C}$. We equip $L(G)$ with the *convolution product* $*$ defined by setting, for $f_1, f_2 \in L(G)$,

$$(f_1 * f_2)(g) = \sum_{h\in G} f_1(gh^{-1})f_2(h) \quad \text{for all } g \in G. \tag{1.1}$$

With the product $*$, the space $L(G)$ becomes an algebra, called the $\mathbb{C}$-*group algebra* of G. Note that $L(G)$ is unital, with unity element δ_{1_G}. Moreover, the map $f \mapsto f^*$, where $f^*(g) := \overline{f(g^{-1})}$ for all $g \in G$, is an involution.

Example 1.2.14 (Regular representations) Let G be a finite group. Then the *left* (resp. *right*) *regular representation* of G is the (unitary) representation $(\lambda_G, L(G))$ (resp. $(\rho_G, L(G))$) defined by setting

$$[\lambda_G(g)f](h) = f(g^{-1}h) \quad (\text{resp. } [\rho_G(g)f](h) = f(hg))$$

for all $g,h \in G$ and $f \in L(G)$.

Exercise 1.2.15 Show that the left (resp. right) regular representation is unitary when $L(G)$ is endowed with the scalar product $\langle\cdot,\cdot\rangle_{L(G)}$ defined by setting

$$\langle f_1, f_2\rangle_{L(G)} := \sum_{g\in G} f_1(g)\overline{f_2(g)}$$

for all $f_1, f_2 \in L(G)$.

Example 1.2.16 (Representations of a cyclic group) Let

$$G = C_n = \{1, a, a^2, \dots, a^{n-1}\} \cong \mathbb{Z}/n\mathbb{Z}$$

denote the *cyclic group* of order n. Consider the primitive nth root of unity $\omega := e^{2\pi i/n}$ and, for $k \in \mathbb{Z}$, let $(\rho_k, \mathbb{C})$ denote the (unitary) representation defined by

$$\rho_k(a^h) = \omega^{kh}\mathrm{Id}_{\mathbb{C}}$$

for all $h = 0, 1, \dots, n-1$. Note that $\rho_k = \rho_{k'}$ if $k \equiv k' \bmod n$ and that $\rho_k \not\sim \rho_{k'}$ if $k \not\equiv k' \bmod n$. In fact, $\widehat{C_n} = \{\rho_k : k = 0, 1, \dots, n-1\}$.

Example 1.2.17 (Two particular representations of the symmetric group) Let $G = S_n = \mathrm{Sym}(\{1, 2, \dots, n\})$ denote the *symmetric group* of degree n, that is the group of all bijective maps (*permutations*) $g\colon \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$.

The *sign representation* of S_n is the one-dimensional representation $(\mathrm{sign}, \mathbb{C})$ defined by

$$\mathrm{sign}(g) = \begin{cases} \mathrm{Id}_{\mathbb{C}} & \text{if } g \in A_n \\ -\mathrm{Id}_{\mathbb{C}} & \text{if } g \in S_n \setminus A_n \end{cases}$$

for all $g \in S_n$, where $A_n \leq S_n$ is the *alternating subgroup* (consisting of all permutations which can be expressed as a product of an even number of transpositions).

Let V be an n-dimensional vector space equipped with a scalar product. Fix an orthonormal basis $\{e_1, e_2, \dots, e_n\} \subset V$. The *permutation representation* of S_n (cf. Definition 1.2.50) is the (unitary) representation (ρ, V) defined by setting

$$\rho(g)e_i = e_{g(i)}$$

for all $g \in S_n$ and $i = 1, 2, \dots, n$.

Exercise 1.2.18 Let $G = S_n$ be the symmetric group of degree n.

Show that the sign representation $(\mathrm{sign}, \mathbb{C})$ is indeed a unitary representation.

Show that the permutation representation (ρ, V) is indeed a unitary representation. Let $W \leq V$ denote the one-dimensional subspace spanned by the vector $e_1 + e_2 + \cdots + e_n$. Show that W is G-invariant. Show that

$$W^{\perp} = \{\sum_{i=1}^{n} \alpha_i e_i : \alpha_i \in \mathbb{C} \text{ and } \alpha_1 + \alpha_2 + \cdots + \alpha_n = 0\}$$

is equal to the linear span of $\{e_i - e_{i-1} : i = 2, 3, \dots, n\}$, and is irreducible. Deduce that $V = W \oplus W^{\perp}$ is the decomposition of V into irreducible components.

Definition 1.2.19 (Commutant) The *commutant* of two G-representations (ρ_1, V_1) and (ρ_2, V_2) is the vector space

$$\mathrm{Hom}_G(V_1,V_2) := \{T : V_1 \to V_2 : T \text{ is linear and } T\rho_1(g) = \rho_2(g)T \text{ for all } g \in G\}.$$

We refer to its elements as to the *intertwiners* of ρ_1 and ρ_2. When $V_1 = V_2 = V$ we denote the *commutant* $\mathrm{Hom}_G(V,V)$ by $\mathrm{End}_G(V)$. It has a natural structure of an algebra.

Exercise 1.2.20 Let (ρ_1, V_1), (ρ_2, V_2), and (ρ, V) be unitary G-representations. Given $T \in \mathrm{Hom}_G(V_1,V_2)$, let $T^* \colon V_2 \to V_1$ denote the adjoint operator. Show that $T^* \in \mathrm{Hom}_G(V_2,V_1)$. Show that the commutant $\mathrm{End}_G(V)$ has a natural structure of a $*$-algebra.

The following is a celebrated, elementary but extremely useful result of Schur.

Lemma 1.2.21 (Schur's lemma) *Let (ρ_1, V_1) and (ρ_2, V_2) be two irreducible G-representations. If $T \in \mathrm{Hom}_G(V_1,V_2)$, then either $T = 0$ or T is an isomorphism (and $\rho_1 \sim \rho_2$).*

Proof The kernel $\ker T \le V_1$ and the image $\operatorname{ran} T \le V_2$ are G-invariant subspaces, and by the irreducibility of ρ_1 and ρ_2 they must be trivial. If $\ker T = \{0\}$, then $\operatorname{ran} T = V_2$ and therefore T is an isomorphism; and if $\ker T = V_1$, then $T \equiv 0$. □

Corollary 1.2.22 *Let (ρ, V) be an irreducible G-representation and consider $T \in \mathrm{End}_G(V)$. Then $T \in \mathbb{C}I_V$.*

Proof Let $\lambda \in \mathbb{C}$ be an eigenvalue of T, so that $T - \lambda I_V$ cannot be an isomorphism. As $T - \lambda I_V \in \mathrm{End}_G(V)$, Schur's lemma (Lemma 1.2.21) ensures that $T - \lambda I_V \equiv 0$, that is, $T = \lambda I_V$. □

Exercise 1.2.23 Let G be a group. Show that if G is abelian and (ρ, V) is a G-representation, then ρ is irreducible if and only if $d_\rho = 1$. Show that, vice versa, if every irreducible G-representation is one-dimensional, then G is abelian.
Hint. For the converse implication, use the following steps:

- A representation (ρ, V) of G is *faithful* provided that $\rho(g) \neq I_V$ for all $g \in G \setminus \{1_G\}$. Show that the regular representations (cf. Example 1.2.14) of G are faithful.
- Apply Theorem 1.2.12 to the left regular representation of G and deduce that for every $g \in G \setminus \{1_G\}$, there exists an irreducible representation (ρ_g, V_g) of G such that $\rho_g(g) \neq I_V$.