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Additive and Exact Categories

This chapter covers the foundations of additive and exact category theory that will be the basis for the rest of the book. After reviewing the basics of additive categories, we develop the most fundamental and crucial consequences of the axioms for Quillen exact categories. These facts about short exact sequences will be used constantly throughout the text. The category of abelian groups will be denoted by **Ab**.

1.1 Additive Categories

Additive categories are truly fundamental structures in homological algebra. They include all abelian categories, such as abelian groups, vector spaces, and modules over a general ring R, not to mention categories of (quasi-coherent) sheaves and their associated categories of chain complexes. Moreover, *exact categories* and *triangulated categories*, both of which play a central role in this book, are special classes of additive categories.

A category $\mathcal{A}$ is called a **preadditive category** if every hom-set, which we will denote by $\mathrm{Hom}_{\mathcal{A}}(A, B)$, is an abelian group and composition of morphisms is bilinear. That is, $f(g + h) = fg + fh$ and $(f + g)h = fh + gh$ whenever these compositions make sense. Alternatively, $\mathcal{A}$ is often called an **Ab**-*category*, for in a technical sense it is a category "enriched" over the category **Ab**.

A morphism of preadditive categories is called an **additive functor**. It is a functor $F : \mathcal{A} \to \mathcal{B}$ between additive categories $\mathcal{A}$ and $\mathcal{B}$ for which each function

$$F_{A,B} : \ \mathrm{Hom}_{\mathcal{A}}(A, B) \to \mathrm{Hom}_{\mathcal{B}}(FA, FB)$$

is an abelian group homomorphism.

It is sometimes a helpful perspective to think of a preadditive category as a "gigantic ring with unities", where the morphisms are the ring elements and composition is the multiplication. Of course not all arrows can be added and/or multiplied, but whenever they can we have properties similar to a ring with unity. Here are some examples of what we have in mind.

- Bilinearity corresponds to the left and right distribute laws.
- The maps 1_A and 0_A act like "unities" and "zeros". For example, composing any morphism with a 0 yields another 0. This follows from bilinearity in the same way that one uses the distributive law to show that a ring element r satisfies the properties $0 \cdot r = r \cdot 0 = 0$.
- Also similar to the usual ring theory proofs, we have the compatibilities

$$(-g)f = -1(gf) = -(gf) = g(-f).$$

- Additive functors are analogous to ring homomorphisms. In fact, for any object $A \in \mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}}(A, A)$ is a ring, called the endomorphism ring of A. If $F \colon \mathcal{A} \to \mathcal{B}$ is an additive functor, then each function

$$F_{A,A} \colon \ \mathrm{Hom}_{\mathcal{A}}(A, A) \to \mathrm{Hom}_{\mathcal{B}}(FA, FA)$$

  is a ring homomorphism.
- In the same way that a monoid is equivalent to a category with one object, a ring with unity is equivalent to a preadditive category with one object.

The simplicity of preadditive categories stems from the fact that finite products and coproducts coincide, if they exist. The first instance of this is the following lemma. Recall that, by definition, an *initial* object I is an object of a category C for which there is exactly one morphism $I \to C$ for each object $C \in C$. The dual notion defines a *terminal* object.

Lemma 1.1 *If an object of a preadditive category $\mathcal{A}$ is either an initial or a terminal object, then that object must be both initial and terminal.*

Before proving the lemma, we note that such an object is called a **zero object**, or a **null object**, and typically denoted by 0. Clearly any such object is unique up to a unique isomorphism. Our proof of the lemma will show that $X = 0$ if and only if $1_X = 0_X$.

Proof If I is initial, then $\mathrm{Hom}_{\mathcal{A}}(I, I)$ must have exactly one element. Hence $1_I = 0_I$. We then can see that I must also be terminal as follows. First, there does exist at least one morphism $A \to I$, namely the zero morphism $A \xrightarrow{0} I$. But given any morphism $f \colon A \to I$, we have $f = 1_I f = 0_I f = 0$. This proves I is terminal. By duality, any terminal object must also be initial. □

A preadditive category need not have a 0 object. For example, any (nonzero) ring R with unity, considered as a preadditive category with one object, doesn't have a 0 object. Another example is the category $\{R^n \mid n \geq 1\}$ of free R-modules of finite rank $n \geq 1$. Note that the hom-sets, $\mathrm{Hom}(R^n, R^m)$, are equivalent to $M_{m\times n}$, the additive group of all $m \times n$ matrices over R.

Exercise 1.1.1 Let $\mathcal{A}$ be a preadditive category with a zero object 0. Show that the zero morphism $(A \xrightarrow{0} B) \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ coincides with the unique composition $A \to 0 \to B$, through the zero object.

A zero object may be viewed as an "empty biproduct". In fact, the amalgamation of initial and terminal objects in preadditive categories generalizes to the fact that finite coproducts coincide with finite products. Here we will only explicitly define biproducts for a given collection of two objects ($n = 2$), but it can be generalized in a straightforward way to any $n \geq 0$ (with $n = 0$ being a null object).

Definition 1.2 Let A and C be objects of a preadditive category $\mathcal{A}$. A **biproduct diagram** for A and C is a diagram of the form

$$A \underset{i_A}{\overset{p_A}{\leftrightarrows}} B \underset{p_C}{\overset{i_C}{\leftrightarrows}} C$$

satisfying (i) $p_A i_A = 1_A$, (ii) $p_C i_C = 1_C$, and (iii) $i_A p_A + i_C p_C = 1_B$. In this case we write $B = A \bigoplus C$, and say that B is the **direct sum**, or **biproduct**, of A and C. We also say that A and C are **direct summands** of B.

An **additive category** is simply a preadditive category $\mathcal{A}$ in which all finite biproducts exist. In particular, this includes the existence of a zero object.

The categorical notion of the kernel of a morphism makes sense in any preadditive category. If the kernel of $f : A \to B$ exists, we denote it by $\ker f$. Note that $\ker f$ is exactly the same thing as an equalizer of the pair

$$A \underset{0}{\overset{f}{\rightrightarrows}} B.$$

The dual notion of cokernel, denoted $\operatorname{cok} f$ when it exists, is the coequalizer of f and 0. The following exercise is fundamental as it states that (i_A, p_C) and (i_C, p_A) form what we call kernel–cokernel pairs.

Exercise 1.1.2 Show $i_A = \ker p_C$, $p_C = \operatorname{cok} i_A$, $i_C = \ker p_A$, and $p_A = \operatorname{cok} i_C$ in any biproduct diagram as in Definition 1.2.

The following exercise asks the reader to prove the fact alluded to previously: Finite coproducts coincide with finite products in preadditive categories, and determine biproducts.

Exercise 1.1.3 Let A and C be objects of a preadditive category $\mathcal{A}$. Then, if they exist, their product $A \prod C$, coproduct $A \coprod C$, and biproduct $A \bigoplus C$, all coincide. More precisely, suppose we have a biproduct diagram as in Definition 1.2. Then $\left(A \bigoplus C, p_A, p_C\right)$ is the product of A and C, and $\left(A \bigoplus C, i_A, i_C\right)$ is the coproduct of A and C. Conversely, any product diagram or coproduct diagram determines a biproduct diagram.

A convenient feature of additive categories is that we may use matrix notation to denote morphisms in and out of a biproduct. Being a product, a morphism $X \to A \bigoplus C$ is equivalent to a matrix $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$ for some $X \xrightarrow{f} A$ and $X \xrightarrow{g} C$. On the other hand, being a coproduct, a morphism $A \bigoplus C \to Y$ is equivalent to a matrix $[\, f \; g \,]$ for some $A \xrightarrow{f} Y$ and $C \xrightarrow{g} Y$. Note that if we are given compositions $X \xrightarrow{f} A \xrightarrow{g} Y$ and $X \xrightarrow{\alpha} C \xrightarrow{\beta} Y$, then we have the morphism $X \xrightarrow{gf+\beta\alpha} Y$ which factors as

$$X \xrightarrow{\left[\begin{smallmatrix} f \\ \alpha \end{smallmatrix}\right]} A \bigoplus C \xrightarrow{[\, g \; \beta \,]} Y,$$

where we use the usual matrix multiplication $[\, g \; \beta \,]\left[\begin{smallmatrix} f \\ \alpha \end{smallmatrix}\right] = gf + \beta\alpha$.

We perhaps should call an object B as in Definition 1.2 the *internal direct sum* of A and C. For conversely, given any two objects A and C, assuming their coproduct (equivalently product) exists we may denote it by $A \bigoplus C$. We have the **canonical split exact sequence**

$$A \underset{\left[\begin{smallmatrix} 1_A \\ 0 \end{smallmatrix}\right]}{\overset{[\, 1_A \; 0 \,]}{\leftrightarrows}} A \bigoplus C \underset{[\, 0 \; 1_C \,]}{\overset{\left[\begin{smallmatrix} 0 \\ 1_C \end{smallmatrix}\right]}{\leftrightarrows}} C \tag{1.1}$$

for A and C, and we call $A \bigoplus C$ the *external direct sum* of A and C.

Lemma 1.3 *Given any biproduct B as in Definition 1.2, the morphism $\left[\begin{smallmatrix} p_A \\ p_C \end{smallmatrix}\right]$: $B \to A \bigoplus C$ is an isomorphism to the external direct sum, with inverse $[\, i_A \; i_C \,]$: $A \bigoplus C \to B$. The isomorphisms are compatible in the sense that they make the entire biproduct diagram for the internal direct sum commute with the biproduct diagram for the canonical split exact sequence of Diagram* (1.1).

Proof We have $[\, i_A \; i_C \,]\left[\begin{smallmatrix} p_A \\ p_C \end{smallmatrix}\right] = i_A p_A + i_C p_C = 1_B$. On the other hand,

$$\left[\begin{smallmatrix} p_A \\ p_C \end{smallmatrix}\right][\, i_A \; i_C \,] = \left[\begin{smallmatrix} p_A i_A & p_A i_C \\ p_C i_A & p_C i_C \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1_A & 0 \\ 0 & 1_C \end{smallmatrix}\right] = 1_{A \bigoplus C}.$$

Moreover, the isomorphisms make all diagrams in sight commute. □

We end this section by clarifying the connection between split exact sequences (i.e. biproduct diagrams) and the notion of *split monics* and *split epics*. We will return to this important idea is Section 1.8.

In any category, a morphism $f\colon A \to B$ is called a **split monomorphism** if it admits a left inverse $r\colon B \to A$, called a **retraction**. That is, we have $rf = 1_A$. We then also say that A is a **retract** of B. The retraction r then satisfies the dual notion of being a split epimorphism. Let us state formally that a morphism $g\colon B \to C$ is called a **split epimorphism** if it admits a right inverse $s\colon C \to B$, called a **section**, and so satisfying $gs = 1_C$. Certainly s is a split monomorphism, and so C is a retract of B. Note that a split monomorphism is indeed a monomorphism since it is easily seen to be left cancellable, while a split epimorphism is an actual epimorphism. In most familiar additive categories split monomorphisms (and split epimorphisms) give rise to biproduct diagrams. The next statement clarifies when this actually happens.

Proposition 1.4 *Let $\mathcal{A}$ be an additive category and let $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ be morphisms. The following are equivalent.*

(1) *We have a biproduct diagram $A \underset{f}{\overset{r}{\leftrightarrows}} B \underset{g}{\overset{s}{\leftrightarrows}} C$. In particular, A and C are direct summands of $B = A \bigoplus C$.*
(2) *The morphism g is a split epimorphism and admits a kernel. In this case, let $f = \ker g$ denote a kernel, and $C \xrightarrow{s} B$ denote a section $gs = 1_C$.*
(3) *The morphism f is a split monomorphism and admits a cokernel. In this case, let $g = \operatorname{cok} f$ denote a cokernel, and $B \xrightarrow{r} A$ denote a retraction $rf = 1_A$.*

Proof In light of Exercise 1.1.2, we clearly have (1) implies (2) and (3). We only prove (2) implies (1) as (3) implies (1) is dual. So suppose $f = \ker g$ and $gs = 1_C$. Then $gsg = g$, which implies $g(1_B - sg) = 0$. By the universal property of f, there exists a unique morphism $r\colon B \to A$ such that $fr = 1_B - sg$. Hence $1_B = fr + sg$, and it only remains to show $rf = 1_A$. For this, we note $frf = (1_B - sg)f = f - s(0) = f = f1_A$. Since f is left cancellable we conclude $rf = 1_A$. □

Exercise 1.1.4 Show that an additive functor between additive categories necessarily preserves null objects (zero objects) and biproducts.

Exercise 1.1.5 Consider a sequence of morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ in an additive category $\mathcal{A}$. Verify that the following statements are equivalent.

(1) $f = \ker g$.

(2) $\begin{bmatrix} f \\ 0 \end{bmatrix}$ is the kernel of $\begin{bmatrix} g & 0 \\ 0 & 1_D \end{bmatrix}: B \bigoplus D \to C \bigoplus D$ for any object D.

(3) $\begin{bmatrix} f & 0 \\ 0 & 1_D \end{bmatrix}: A \bigoplus D \to B \bigoplus D$ is a kernel of $[\, g \;\; 0 \,]$ for any object D.

Dually we have $g = \operatorname{cok} f$ if and only if $[\, g \;\; 0 \,]$ is the cokernel of $\begin{bmatrix} f & 0 \\ 0 & 1_D \end{bmatrix}$: $A \bigoplus D \to B \bigoplus D$ for any object D, if and only if $\begin{bmatrix} g & 0 \\ 0 & 1_D \end{bmatrix}: B \bigoplus D \to C \bigoplus D$ is a cokernel of $\begin{bmatrix} f \\ 0 \end{bmatrix}$ for any object D.

1.2 Pushouts and Pullbacks in Additive Categories

The point of this brief section is simply to record some useful facts about pushouts and pullbacks in additive categories. These are standard results and are straightforward to prove, so much of the proofs will be left as exercises. Let $\mathcal{A}$ be an additive category throughout.

A key observation is that pushouts and pullbacks are essentially equivalent to cokernels and kernels, respectively. Indeed a square as shown is equivalent to a sequence of morphisms as shown:

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ B & \xrightarrow{g'} & D \end{array} \qquad\qquad A \xrightarrow{\begin{bmatrix} f \\ -g \end{bmatrix}} B \bigoplus C \xrightarrow{[\, g' \;\; f' \,]} D. \qquad (1.2)$$

Moreover, the square commutes if and only if the sequence is a null sequence. By a **null sequence** we mean any pair of composable morphisms, $A \xrightarrow{f} B \xrightarrow{g} C$, such that $gf = 0$. It is easy to prove the following lemma by translating between the universal properties involved.

Lemma 1.5 *Consider the square and corresponding sequence given in Diagrams* (1.2).

(1) *The square is a pullback if and only if* $\begin{bmatrix} f \\ -g \end{bmatrix}$ *is the kernel of* $[\, g' \;\; f' \,]$.

(2) *The square is a pushout if and only if* $[\, g' \;\; f' \,]$ *is the cokernel of* $\begin{bmatrix} f \\ -g \end{bmatrix}$.

The single negative sign, appearing as $-g$ in Diagram (1.2)*, may be placed in any of the four possibilities within* $\begin{bmatrix} f \\ g \end{bmatrix}$ *and* $[\, g' \;\; f' \,]$.

We ask the reader to also prove the following lemma.

Lemma 1.6 *Suppose we have a commutative diagram in $\mathcal{A}$:*

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C', \end{array}$$

where both rows are null sequences.

(1) *If $f = \ker g$ and γ and f' are each monomorphisms, then the left-hand square is necessarily a pullback.*
(2) *If $g' = \operatorname{cok} f'$ and α and g are each epimorphisms, then the right-hand square is necessarily a pushout.*

Next we will prove that pushout and pullback squares may be composed, or "pasted" together.

Lemma 1.7 (Composing and Factoring Pushout Squares) *Assume the following left square is a pushout and the entire diagram commutes:*

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ D & \xrightarrow[f']{} & E & \xrightarrow[g']{} & F. \end{array}$$

Then the right square is a pushout if and only if the outer rectangle is a pushout.

Of course there is also a dual statement concerning pullback squares.

Proof First let us assume that the outer rectangle is a pushout and prove that the right square is a pushout. So assume we are given morphisms $C \xrightarrow{s} Z$ and $E \xrightarrow{t} Z$ such that $t\beta = sg$. Then $t\beta f = sgf \implies (tf')\alpha = sgf$. So since the outer rectangle is a pushout we get a unique $F \xrightarrow{\xi} Z$ such that

$$\begin{aligned} \xi g' f' &= tf', \\ \xi\gamma &= s. \end{aligned} \qquad (*)$$

We wish to show that ξ also uniquely satisfies the set of equations:

$$\begin{aligned} \xi g' &= t, \\ \xi\gamma &= s. \end{aligned} \qquad (**)$$

But the uniqueness portion is obvious since the Equations $(**)$ imply Equations $(*)$. So we only need to show $\xi g' = t$. Note that since the left square is a pushout, and we have the "impostor" square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle 0} \\ D & \xrightarrow{0} & Z, \end{array}$$

it is enough to show that $(\xi g'-t)f' = 0$ and $(\xi g'-t)\beta = 0$. Clearly, $(\xi g'-t)f' = 0$ by Equations $(*)$. Now we compute:

$$\begin{aligned}(\xi g' - t)\beta &= \xi g'\beta - t\beta \\ &= \xi\gamma g - t\beta \\ &= sg - t\beta, \text{ by Equations } (**) \\ &= 0.\end{aligned}$$

Conversely, assume the right square is a pushout and that $C \xrightarrow{u} W$ and $D \xrightarrow{v} W$ are morphisms such that $u(gf) = v\alpha$. Using the pushout property of the left square, followed by the pushout property of the right square, we easily find a morphism $F \xrightarrow{\xi} W$ such that $\xi(g'f') = v$ and $\xi\gamma = u$. To prove that the outer rectangle is a pushout it is only left to show that ξ is the unique morphism with this property. So let us assume that $F \xrightarrow{\xi'} W$ also satisfies $\xi'(g'f') = v$ and $\xi'\gamma = u$ and our goal is to show that $\xi' - \xi = 0$. By the universal property of the right pushout square, it is enough to show $(\xi' - \xi)g' = 0$ and $(\xi' - \xi)\gamma = 0$. It is immediate that $\xi'\gamma - \xi\gamma = 0$, so we just need to show $\xi'g' - \xi g' = 0$. But by the universal property of the left pushout square, it is enough to show $(\xi'g' - \xi g')f' = 0$ and $(\xi'g' - \xi g')\beta = 0$. It is immediate that $(\xi'g' - \xi g')f' = 0$ and for the second equation we have $\xi'g'\beta - \xi g'\beta = \xi'\gamma g - \xi\gamma g = ug - ug = 0$. We conclude that the outer rectangle is also a pushout. □

We have one final lemma that is fundamental to pullback and pushout arguments in additive and exact categories.

Lemma 1.8 *Suppose we have a commutative diagram in $\mathcal{A}$:*

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ B & \xrightarrow{g'} & D. \end{array}$$

(1) *Assume the square is a pullback. If $k = \ker g$ exists, then $fk = \ker g'$. On the other hand, if $k' = \ker g'$, then $k' = fk$ for some unique morphism k and $k = \ker g$.*

(2) *Assume the square is a pushout. If* $c' = \operatorname{cok} g'$ *exists, then* $c'f' = \operatorname{cok} g$*. On the other hand, if* $c = \operatorname{cok} g$*, then* $c = c'f'$ *for some unique morphism* c' *and* $c' = \operatorname{cok} g'$*.*

Exercise 1.2.1 Prove Lemma 1.5.

Exercise 1.2.2 Prove Lemma 1.6.

Exercise 1.2.3 Prove Lemma 1.8.

1.3 Exact Categories: The Axioms

We now introduce the notion of an exact category which will be the categorical foundation for our theory of abelian model structures. Interestingly enough, exact categories were also introduced by Quillen, though independently of his introducing model categories. Loosely, an exact category is an additive category $\mathcal{A}$ along with a distinguished class of kernel–cokernel pairs satisfying several closure axioms. An abelian category $\mathcal{A}$, along with the class of all short exact sequences in $\mathcal{A}$, is the primary and motivating example of an exact category. Extension closed full subcategories of abelian categories are another prime example. Several specific examples will appear in the exercises and elsewhere throughout this book.

First, by a **kernel–cokernel pair** we mean an ordered pair (i, p) of composable morphisms $A \xrightarrow{i} B \xrightarrow{p} C$ such that $i = \ker p$ and $p = \operatorname{cok} i$. Note that such an i is necessarily monic and such a p is necessarily epic. (We will use the term *monic* interchangeably with *monomorphism*, and *epic* interchangeably with *epimorphism*.) A morphism from a kernel–cokernel pair (i, p) to another (i', p') is a triple of morphisms making the diagram commute:

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{p} & C \\ \downarrow & & \downarrow & & \downarrow \\ A' & \xrightarrow{i'} & B' & \xrightarrow{p'} & C'. \end{array}$$

Such a morphism is an isomorphism from (i, p) to (i', p') when all three morphisms are isomorphisms.

To motivate the axioms for exact categories, let us recall some well-known facts about abelian categories, (which will appear in more detail in Section 1.9). A defining aspect of abelian categories is that every monic i is the kernel of its cokernel and therefore is part of a kernel–cokernel pair (i, p). The dual is true as well; each epic appears as a p in some kernel–cokernel pair. So in the abelian case, kernel–cokernel pairs are equivalent to the usual notion of *short exact sequences*, and they satisfy a number of useful properties. In particular, the composite of two monics is again a monic and hence the first component of another short exact sequence. Also, the pushout of any morphism along a monic is again a monic. The dual statements hold for epics, and it turns out that a great deal of pure homological algebra follows from these formal properties. Extracting them leads to Quillen's notion of an exact category.

Definition 1.9 Let $\mathcal{A}$ be an additive category and $\mathcal{E}$ be a class of kernel–cokernel pairs in $\mathcal{A}$. We say that $\mathcal{E}$ is an **exact structure** on $\mathcal{A}$, and that $(\mathcal{A}, \mathcal{E})$ is an **exact category**, if $\mathcal{E}$ satisfies the following axioms. The axioms utilize this standard terminology: If (i, p) is a kernel–cokernel pair in $\mathcal{E}$, then we say that (i, p) is a **short exact sequence**, i is an **admissible monic**, and p is an **admissible epic**.

(Ex1) (*Replete*) $\mathcal{E}$ is closed under isomorphisms.

(Ex2) (*Split Exact Sequences*) $\mathcal{E}$ contains the canonical split exact sequence

$$A \xrightarrow{\left[\begin{smallmatrix} 1_A \\ 0 \end{smallmatrix}\right]} A \bigoplus C \xrightarrow{[\,0 \;\; 1_C\,]} C$$

for any given pair of objects A and C.

(Ex3) (*Pushouts and Pullbacks*) The pushout of an admissible monic along any morphism exists and is again an admissible monic. Dually, the pullback of an admissible epic along any morphism exists and is again an admissible epic.

(Ex4) (*Compositions*) The class of admissible monics is closed under composition. Dually, the class of admissible epics is closed under composition.

It is standard to use the symbol $\rightarrowtail$ to denote an admissible monic and $\twoheadrightarrow$ to denote an admissible epic. A short exact sequence is denoted by $A \rightarrowtail B \twoheadrightarrow C$. In the literature, the terms *conflation*, *inflation*, and *deflation*, are also sometimes used for *short exact sequence*, *admissible monic*, and *admissible epic*.

Note that a morphism $f : A \to B$ is an isomorphism if and only if $f : A \rightarrowtail B$ is an admissible monic with $\operatorname{cok} f = 0$, if and only if $f : A \twoheadrightarrow B$ is an admissible epic with $\ker f = 0$. This can be seen from Axiom **(Ex2)** and Proposition 1.4.

Given any additive category $\mathcal{A}$, taking $\mathcal{E}$ to be the class of all split exact sequences determines an exact category $(\mathcal{A}, \mathcal{E})$.

Exercise 1.3.1 Let $\mathcal{A}$ be any additive category. Let $\mathcal{E}$ be the class of all split exact sequences. Show that $(\mathcal{A}, \mathcal{E})$ is an exact category. Note Exercise 1.1.2.

Note that the exact category axioms **(Ex1)**–**(Ex4)** are self-dual. So as a matter of convenience, each result we prove has an equally valid dual.