Introduction

As explained in the Preface, this book is intended for graduate students, with a good working knowledge of quantum field theory, like from a good two-semester course, but which can have very little, or even no, knowledge of general relativity, since these are (quickly) described in the first five chapters. The other 10 chapters of Part I describe the formalism of supergravity, and the 16 chapters of Part II describe applications.

For the formalism part, I describe the most important issues, the on- and off-shell supergravity, the covariant formulation, superspace and coupling to matter, higher dimensions, and extended supersymmetry methods. Besides the standard component formalism, a large part of the presentation is devoted to superspace and coset theory, as well as superspace constraints in terms of torsions and curvatures.

For the applications part, I describe more standard ones such as T-dualities (though generalized to solution-generating techniques such as non-Abelian T-duality, TsT, and $O(d,d)$ transformations), extremal p-brane solutions and their susy algebras, the attractor mechanism, AdS/CFT and gravity duals, inflation with supergravity, compactification of low-energy string theory, and toward embedding the Standard Model in supergravity. In addition, I describe less standard ones such as U-duality, susy and integrable deformations, Penrose limits, supergravity on the string worldsheet, superembeddings, supergravity no-go theorems, and Witten's positive energy theorem.

My goal being to equip the graduate student with the tools and knowledge in the broad field of supergravity, I present a broad range of methods and applications, but I do not make a comprehensive analysis of each of them, rather I focus on the essentials.

After each chapter, I summarize a set of "Important concepts to remember," and present four exercises whose solution is meant to clarify the concepts in the chapter.

PART I

FORMALISM

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1 Introduction to general relativity 1: Kinematics and Einstein equations

In this chapter, I will give a lightning review of the basics of general relativity, from how it is built, to its kinematics, and finally to its dynamics, given by the Einstein equation.

1.1 Intrinsically curved spacetime and the geometry of general relativity

I will start with the need for and meaning of intrinsically curved spacetime, which will lead us to the geometry of general relativity.

But since general relativity is a generalization of special relativity, I will review its basic ideas in order to be able to generalize it.

1.1.1 Special relativity

Special relativity was developed as a result of the experimental observation that the speed of light in a vacuum is equal to a constant in all inertial reference frames, where the constant can be put to 1, so that $c = 1$. This then becomes a postulate of special relativity.

As a result, we find that the line element, or the infinitesimal distance between two points, must be taken in *spacetime*, not just in space, in order to be invariant under transformations of coordinates between any inertial reference frames. This invariant distance is then

$$ds^2 = -dt^2 + d\vec{x}^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \tag{1.1}$$

where $\eta_{\mu\nu} = \text{diag}(-1, 1, ..., 1)$ is the Minkowski metric. This now takes the role of the invariant length $d\vec{x}^2$ in Newtonian physics, which is invariant under rotations of space at a given time.

The symmetry group that leaves ds^2 invariant is $SO(1, 3)$, or in a general dimension $SO(1, d-1)$, called the Lorentz group. It is a generalization of the group $SO(d-1)$ of spatial rotations that leaves $d\vec{x}^2$ invariant. The corresponding Lorentz transformations are linear transformations of the coordinates that generalize rotations, $x'^i = \Lambda^i{}_j x^j$, where $\Lambda \in SO(d-1)$, which leaves invariant $d\vec{x}^2$. Now, instead, we have

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu; \quad \Lambda^\mu{}_\nu \in SO(1, 3), \tag{1.2}$$

which leaves invariant ds^2.

In conclusion, special relativity is defined by the following statement: Physics is Lorentz invariant or covariant (under $SO(1, d-1)$ transformations). It replaces the statement of Newtonian or Galilean physics that physics is invariant under the Galilean group, of spatial rotations, with no action on time.

1.1.2 General relativity

Now to define general relativity, we need to consider the most general line element

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu, \tag{1.3}$$

where $g_{\mu\nu}(x)$ is a symmetric matrix of functions called "the metric." By extension, sometimes one calls the corresponding ds^2 the metric. Moreover, consider here that x^μ make up an arbitrary parametrization of spacetime, that is, are arbitrary coordinates on a manifold.

Example 1 ***S^2* in angular coordinates.** To understand the notation, consider the usual case of a two-dimensional sphere, described in terms of angles. Then the line element is

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2, \tag{1.4}$$

so $x^\mu = (\theta, \phi)$. Then it follows that $g_{\theta\theta} = 1, g_{\phi\phi} = \sin\theta$, and $g_{\theta\phi} = 0$.

Example 2 ***S^2* as an embedding in three-dimensional Euclidean space.** We can describe the sphere also by embedding it in three Euclidean dimensions, meaning as we usually understand it, as an object in three-dimensional space, with the metric

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 \tag{1.5}$$

defined by the constraint

$$x_1^2 + x_2^2 + x_3^2 = R^2. \tag{1.6}$$

Differentiating the constraint, we obtain

$$\begin{aligned} &2(x_1dx_1 + x_2dx_2 + x_3dx_3) = 0 \\ &\Rightarrow dx_3 = -\frac{x_1dx_1 + x_2dx_2}{x_3} = -\frac{x_1}{\sqrt{R^2 - x_1^2 - x_2^2}}dx_1 - \frac{x_2}{\sqrt{R^2 - x_1^2 - x_2^2}}dx_2, \end{aligned} \tag{1.7}$$

and by substituting it back into the Euclidean metric, we obtain the *induced metric on the S^2*,

$$\begin{aligned} ds^2_{\text{induced}} &= dx_1^2\left(1 + \frac{x_1^2}{R^2 - x_1^2 - x_2^2}\right) + dx_2^2\left(1 + \frac{x_2^2}{R^2 - x_1^2 - x_2^2}\right) + 2dx_1dx_2\frac{x_1x_2}{R^2 - x_1^2 - x_2^2} \\ &= g_{\mu\nu}(x^\rho)dx^\mu dx^\nu. \end{aligned} \tag{1.8}$$

This was an example of a curved d-dimensional space obtained by embedding it into a flat (Euclidean or Minkowski) $(d+1)$-dimensional space. We can ask: Is this always possible? The answer is no. To see this, first note that $g_{\mu\nu}$ is a symmetric matrix, with $d(d+1)/2$ arbitrary components. Then, the general coordinate transformations $x'^\mu = x'^\mu(x^\rho)$ correspond to d arbitrary functions, which can be used to put d components to zero, thus remaining $d(d-1)/2$ independent components of $g_{\mu\nu}$. On the other hand, if we were to embed the manifold M^d into $(d+1)$-dimensional Euclidean space E^{d+1}, there would be a unique coordinate x^{d+1} written as a function of the others, $x^{d+1} = x^{d+1}(x^\rho)$, as in the example of the sphere. We see that $d(d-1)/2 = 1$ is true only in the particular case of $d = 2$.

We note here that general coordinate transformations $x'^\mu = x'^\mu(x^\rho)$ act on *the fields* $g_{\mu\nu}(x)$, that is, the functions of spacetime, allowing us to fix their d components, so we have a redundancy similar to the one in gauge transformations in field theory; thus, we can say that general coordinate invariance is a kind of gauge invariance. We will see that we can turn this observation into a useful tool later on.

If we cannot always embed the manifold M^d into $(d+1)$-dimensional space, can we do it by adding more extra dimensions? At first sight, we would say yes, perhaps by adding not 1, but $d(d-1)/2$ dimensions in general. But actually, the situation is worse than that: We also need to make, *case by case*, a discrete choice of the *signature* of the space into which we are embedding a manifold.

Even in the simplest case of two-dimensional surfaces, we need to make this choice: Do we embed two-dimensional surfaces into a 3-dimensional Euclidean space like in the case of the sphere, with signature $(+,+,+)$, or into a three-dimensional Minkowski space, with signature $(-,+,+)$? Note that, since the multiplication of the metric by a sign changes only the convention, these are the only possibilities in three dimensions (the $(-,-,-)$ and $(-,-,+)$ ones are related by multiplication by a sign).

The example of embedding Lobachevsky space into Minkowski space is a famous one, defined by the constraint

$$x^2 + y^2 - z^2 = -R^2. \tag{1.9}$$

Lobachevsky space cannot be embedded into Euclidean space but only into Minkowski space with the metric

$$ds^2 = dx^2 + dy^2 - dz^2, \tag{1.10}$$

with the minus sign in the same place as in the constraint. We might think that this is because the signature on the two-dimensional Lobachevsky space is Minkowski, $(-,+)$ (equivalent to $(+,-)$), but that is wrong also: The signature on the space is two-dimensional Euclidean, so $(+,+)$ or equivalently, $(-,-)$. That is, $\det g_{\mu\nu} > 0$ and not < 0. Indeed, by differentiating the constraint, like for the sphere, we obtain

$$dz = \frac{xdx + ydy}{z} = \frac{xdx + ydy}{\sqrt{R^2 + x^2 + y^2}}, \tag{1.11}$$

and by replacing in the Minkowski metric, we obtain the induced metric on the Lobachevsky space,

$$ds^2_{\text{induced}} = dx^2 + dy^2 + \frac{(xdx + ydx)^2}{R^2 + x^2 + y^2} \equiv g_{\mu\nu}dx^\mu dx^\nu, \tag{1.12}$$

which is positive definite, so $\det g_{\mu\nu} > 0$.

Finally, this means that even two-dimensional surfaces of Euclidean signature can be embedded in three dimensions, but either in Euclidean or in Minkowski ones, depending on the surface. In higher dimensions, the number of choices for the signature becomes even larger, so defining spaces by embedding is possible, but very complicated and not very useful.

Instead, we must consider spaces as intrinsically curved, without embedding, and that in turn leads to non-Euclidean, Riemannian, geometry. This observation was believed to be first made by Gauss, who tried to measure if our space is actually curved (but failed, of course; on scales of even kilometers, space is flat to a very high accuracy).

In curved spaces, to define geometry, we must first define the analog of "straight lines" of Euclidean geometry, which are the geodesics, also defined as lines of the shortest distance $\int_a^b ds$ between two points a and b. In non-Euclidean geometry, a triangle made by two geodesics has the sum of its inner angles, $\alpha + \beta + \gamma \neq \pi$. In Euclidean geometry, of course, the sum is *equal to* π by a theorem.

On spaces like S^2 of "positive curvature," $R > 0$, we have $\alpha + \beta + \gamma > \pi$, as we can easily see in the following example: Consider a triangle made by two meridian lines starting from the North Pole and ending on an Equator line. The meridian lines with the Equator line make $\pi/2$ each, so $\alpha + \beta + \gamma > \pi$.

But that is not the only possibility. On a space like Lobachevsky space, we can check that $\alpha + \beta + \gamma < \pi$, and we call this a space of "negative curvature," $R < 0$. We will see in Section 1.3 what $R < 0$ and $R > 0$ means.

In conclusion, we see that for general relativity, we will need intrinsically curved spacetimes, with non-Euclidean geometry, with a general metric $g_{\mu\nu}(x)$, and acted upon by general coordinate transformations that act as gauge transformations.

1.2 Einstein's theory of general relativity

Einstein thought of defining general relativity in order to modify Newton's gravity at high gravitational acceleration $\vec{g}$ and high velocity $\vec{v}$ in order to make it compatible with special relativity. The need for that arose also because of experimental results: The deflection of light by the Sun using only special relativity is a factor of 1/2 off the actual result.

The construction of general relativity was based on two physical assumptions:

(1) **Gravity is geometry**

That is, matter follows geodesics (paths of shortest distance) in curved spacetimes, and to us, it appears as the effect of gravity.

Pictorially, consider a planar rubber sheet and put a heavy ball at a point on it: It will curve the sheet locally. Then, when throwing a light ball on the sheet, the local disturbance deflects it (think of a golfer doing a putt and the golf ball just missing the hole). Of course, this is just a pictorial way of describing the phenomenon; otherwise, it is a cheat: The sheet curves because of the terrestrial gravity it feels, and the curvature is only of space, not of spacetime. But this is a nice way of viewing what happens.

(2) **Matter sources gravity**
This means matter generates the gravitational field that is equated with the curvature of the geometry of spacetime from the first assumption.

These two physical assumptions were then translated into two physical principles with a mathematical formulation, defining the *kinematics* of general relativity, plus one equation for the dynamics, that is, Einstein's equation.

(A) Physics is invariant (or, more generally, covariant) under general coordinate transformations, which generalizes the Lorentz invariance or covariance in the case of special relativity.

For a general coordinate transformation $x'^\mu = x'^\mu(x^\nu)$, we obtain

$$ds^2 = g_{\rho\sigma}(x)dx^\rho dx^\sigma = g'_{\mu\nu}dx'^\mu dx'^\nu, \tag{1.13}$$

giving the transformation rules for the field $g_{\mu\nu}$ (thought of as a field in spacetime),

$$g'_{\mu\nu}(x') = g_{\rho\sigma}(x)\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}. \tag{1.14}$$

This transformation is like a gauge invariance, and physics must be invariant or covariant with respect to it.

(B) The equivalence principle.

In Newtonian theory, there are *a priori* two masses: one is the inertial mass m_i, appearing in Newton's law of force, that is, $\vec{F} = m_i\vec{a}$, and the other is the gravitational mass m_g, appearing in Newton's gravitational law, that is, $\vec{F}_G = m_g\vec{g}$.

The equality of the two masses is the mathematical form of the equivalence principle, that is,

$$m_i = m_g. \tag{1.15}$$

In more physical terms, we say that "there is no difference between gravity and local acceleration." We can also explain this using Einstein's *gedanken (thought) experiment*. Consider a person inside a freely falling elevator with no windows. Then, by performing local experiments inside the elevator, the person cannot distinguish between being weightless and being inside a freely falling elevator. Of course, the *locality* condition is important, because if one is allowed to probe large regions of space, then he or she will note that there are tidal forces – gravity acting at different points in different directions (all pointing toward the center of the Earth). Also, locality in time is important; otherwise, eventually the elevator will hit the hard surface of the Earth, ending the experiment.

On the basis of the above principles, we now turn to constructing the kinematics of general relativity.

First, consider an infinitesimal general coordinate transformation, $x'^\mu = x^\mu - \xi^\mu$, with ξ^μ small, and we want to describe it as a gauge transformation. Then,

$$\begin{aligned} g'_{\mu\nu}(x^\lambda - \xi^\lambda) &= (\delta^\rho_\mu + \partial_\mu\xi^\rho)(\delta^\sigma_\nu + \partial_\nu\xi^\sigma)g_{\rho\sigma}(x) \\ &= g'_{\mu\nu}(x^\lambda) - (\partial_\lambda g'_{\mu\nu}(x))\xi^\lambda, \end{aligned} \tag{1.16}$$

where in the first equality, we used the transformation law of $g_{\mu\nu}$, and in the second equality, we used the Taylor expansion.

Equating the two, we obtain

$$\delta g_{\mu\nu}(x) \equiv g'_{\mu\nu}(x) - g_{\mu\nu}(x) \simeq \xi^\lambda \partial_\lambda g'_{\mu\nu}(x) + (\partial_\mu \xi^\rho) g_{\rho\nu}(x) + (\partial_\nu \xi^\sigma) g_{\mu\sigma}(x)$$
$$\simeq \xi^\lambda \partial_\lambda g_{\mu\nu}(x) + (\partial_\mu \xi^\rho) g_{\rho\nu}(x) + (\partial_\nu \xi^\rho) g_{\mu\rho}(x). \quad (1.17)$$

In formula (1.17), the first term was from the Taylor expansion, so it is just a translation, while the last two terms correspond to a generalized gauge transformation with parameter ξ^ρ instead of the usual α of gauge theory (with $\delta A_\mu = \partial_\mu \alpha$). Since there are two indices on $g_{\mu\nu}$, unlike the case of A_μ, there are two terms, one with ∂_μ and the other with ∂_ν, and the extra metric is needed in order to lower the index on ξ^ρ.

Note that in the global case (with ξ^ρ independent of position), there is only the translation term. Therefore, we can say that *general coordinate transformations are a local version of translations*, and moreover, *General relativity is a "gauge theory of translations."*

1.3 Kinematics

We now move on to defining kinematics per se. We first ask: What is a good variable that corresponds to A_μ in our gauge theory analogy? And correspondingly, what is the respective field strength $F_{\mu\nu}$?

Our first guess would be the metric $g_{\mu\nu}$ itself. We saw that it has $(d(d-1)/2)$-*independent* components (or degrees of freedom, off-shell). However, we know that locally (in a small enough neighborhood), every space looks flat (which in our case means locally Minkowski). In mathematical terms, locally we can always find coordinates such that

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \mathcal{O}(x^2). \quad (1.18)$$

This means also that locally we can define Lorentz transformations, and so there is an $SO(1,3)$ (or $SO(1,d-1)$ in general dimension) invariance, called the *local (x-dependent) Lorentz invariance*.

In any case, this means that $g_{\mu\nu}$ is not a good measure of the curvature of space, but also not quite like the gauge field A_μ either, since A_μ can locally be put to 0, whereas $g_{\mu\nu}$ can only be put to $\eta_{\mu\nu}$.

To understand better what happens, defining general relativity tensors through a simple generalization of special relativity tensors, we have:

– Contravariant tensors A^μ, that are the objects that transform as dx^μ,

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu \Rightarrow A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu. \quad (1.19)$$

– Covariant tensors B_μ that are the objects that transform as ∂_μ,

$$\partial'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu \Rightarrow B'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} B_\nu. \quad (1.20)$$

– Mixed tensors that transform as products, for example,

$$T'^{\mu}{}_{\nu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} T^{\rho}{}_{\sigma}(x), \tag{1.21}$$

and with an obvious generalization to $T^{\mu_1,\dots,\mu_n}_{\nu_1,\dots,\nu_m}$.

Given these definitions, we turn back to the question of what is a good analog of the gauge field A_μ? We can now rephrase this question. Since in gauge theory the covariant derivative $D_\mu = \partial_\mu - iA_\mu$ transforms covariantly, that is, like a covariant vector, we can ask the same question in general relativity as follows: How do we construct a gravitationally covariant derivative?

Since, as we saw, the local Lorentz group is $SO(1, d-1)$, and this is in some sense the gauge group we are looking for, we note that for an $SO(p, q)$ group, the adjoint representation, for the gauge field, is written in terms of the fundamental indices a, b as (ab) (antisymmetric in them), so the gauge covariant derivative on a generic field in the fundamental representation, ϕ^a, is (lowering one index b on the gauge field to have a match with the general relativity construction)

$$D_\mu \phi^a = \partial_\mu \phi^a + (A^a{}_b)_\mu \phi^b. \tag{1.22}$$

In our case, we define something similar to that, with the only difference being that we identify fundamental gauge and spacetime indices, and write for the gravitationally covariant derivative of a contravariant tensor (so that the index is up, just like a on ϕ^a)

$$D_\mu T^\nu = \partial_\mu T^\nu + (\Gamma^\nu{}_\sigma)_\mu T^\sigma, \tag{1.23}$$

where the object $\Gamma^\nu{}_{\sigma\mu}$ is called the "Christoffel symbol," and in Equation (1.23), we put brackets around Γ, just like for the gauge field, but we did not need to, since the gauge and spacetime indices are the same. This object is then the "gauge field of gravity" that we were looking for.

We can easily generalize its action on tensors, by taking into account the position of the indices (only the sign in front is not defined this way), so that

$$D_\mu T^\rho{}_\nu = \partial_\mu T^\rho{}_\nu + \Gamma^\rho{}_{\sigma\mu} T^\sigma{}_\nu - \Gamma^\sigma{}_{\mu\nu} T^\rho{}_\sigma. \tag{1.24}$$

To calculate $\Gamma^\mu{}_{\nu\rho}$ in terms of the metric $g_{\mu\nu}$, we consider the following: If $\Gamma^\mu{}_{\nu\rho}$ is a gauge field, then it should be possible to put it locally to zero by a general coordinate transformation (a gauge transformation), when the space becomes locally flat. At the same time, we saw that $g_{\mu\nu}$ is locally $\eta_{\mu\nu}$. Then

$$D_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^\sigma{}_{\nu\rho} g_{\sigma\rho} - \Gamma^\sigma{}_{\rho\mu} g_{\sigma\nu} = 0 \tag{1.25}$$

locally, but we saw that a tensor transforms by multiplication under general coordinate transformations, so it must be that the result is 0 globally as well (in any coordinate system).

This is an equation whose unique solution is

$$\Gamma^\mu{}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho} \right). \tag{1.26}$$

The proof of this is left as an exercise. Note that here we define the inverse metric $g^{\mu\nu}$ as the matrix inverse of $g_{\nu\rho}$, so $g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$.

Further, we define the Riemann tensor as the analog of the field strength of the $SO(p,q)$ gauge field, $F^{ab}_{\mu\nu}$, namely, since

$$F^{ab}_{\mu\nu} = \partial_\mu A^{ab}_\nu - \partial_\nu A^{ab}_\mu + A^{ac}_\mu A^{cb}_\nu - A^{ac}_\nu A^{cb}_\mu, \tag{1.27}$$

it follows that we can define

$$(R^\mu{}_\nu)_{\rho\sigma}(\Gamma) = \partial_\rho(\Gamma^\mu{}_\nu)_\sigma - \partial_\sigma(\Gamma^\mu{}_\nu)_\rho + (\Gamma^\mu{}_\lambda)_\rho(\Gamma^\lambda{}_\nu)_\sigma - (\Gamma^\mu{}_\lambda)_\sigma(\Gamma^\lambda{}_\nu)_\rho. \tag{1.28}$$

Here we have put brackets around the "gauge indices" to make the analogy with the gauge case more obvious, but, as in the case of the Christoffel symbol, this is not necessary, since gauge and spacetime indices are the same now.

Unlike the gauge case, now we can define the contractions of the Riemann tensor as the Ricci tensor,

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}, \tag{1.29}$$

and as the Ricci scalar,

$$R = R_{\mu\nu} g^{\mu\nu}. \tag{1.30}$$

Finally, the Ricci scalar, by virtue of being a scalar, is invariant under general coordinate transformations, so it is a true invariant measure of the curvature of space at a point, the object we were looking for. In particular, when we said that the sphere was an object of positive curvature $R > 0$ and the Lobachevsky space of negative curvature $R < 0$, we were referring to the Ricci scalar.

The symmetry properties of the Riemann tensor are as follows. First, there are a number of properties that are obvious from the gauge field strength analogy:

1. Since for a gauge field we have the Bianchi identity $(D_{[\mu}F_{\nu\rho]})^a = 0$, we now also have the gravitational Bianchi identity

$$D_{[\lambda}(R^\mu{}_\nu)_{\rho\sigma]} = 0, \tag{1.31}$$

where antisymmetry only acts on $[\lambda\rho\sigma]$.

2, 3. From the antisymmetry of the spacetime indices of the field strength, and of the fundamental indices in the adjoint of $SO(p,q)$, we have (note that we have lowered the first index with a metric on the Riemann tensor for simplicity)

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho}. \tag{1.32}$$

4. Not a symmetry property but the action on a tensor is defined through two covariant derivatives. Since for a gauge field we have $[D_\mu, D_\nu] = F_{\mu\nu}$, which can act on tensors, we now have

$$[D_\mu, D_\nu]T_\rho = -(R^\sigma{}_\rho)_{\mu\nu}T_\sigma = R_{\rho\sigma\mu\nu}T^\sigma. \tag{1.33}$$

5, 6. But then, we have other properties that are not obtained this way, and we must check them from the definition of the Riemann tensor:

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}, \quad R_{\mu[\nu\rho\sigma]} = 0. \tag{1.34}$$