Quantum Geometry, Matrix Theory, and Gravity

Building on mathematical structures familiar from quantum mechanics, this book provides an introduction to quantization in a broad context before developing a framework for quantum geometry in Matrix theory and string theory. Taking a physics-oriented approach to quantum geometry, that framework helps explain the physics of Yang–Mills-type matrix models, leading to a quantum theory of spacetime and matter. This novel framework is then applied to Matrix theory, which is defined through distinguished maximally supersymmetric matrix models related to string theory. A mechanism for gravity is discussed in depth, which emerges as a quantum effect on quantum spacetime within Matrix theory. Using explicit examples and exercises, readers will develop a physical intuition for the mathematical concepts and mechanisms. This book will benefit advanced students and researchers in theoretical and mathematical physics, and provides a useful resource for physicists and mathematicians interested in the geometrical aspects of quantization in a broader context.

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To Fiona

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Preface

This book explores physical and mathematical aspects of quantum geometry and matrix models in the context of fundamental physics. This is motivated by efforts to incorporate gravity into a comprehensive quantum theory, which is expected to entail some quantization of spacetime and geometry.

In the first part of this book, a mathematical framework for quantum geometry is developed, using the mathematical concepts of quantum mechanics to describe spacetime and geometry. Gauge theory on quantum spaces is defined through matrix models, which lead to dynamical quantum geometries naturally interpreted in terms of gravity. Consistency of the quantum theory then leads to Matrix theory, which is defined in terms of a distinguished maximally supersymmetric matrix model related to string theory. Matrix theory is considered as a possible foundation for a quantum theory of fundamental interactions including gravity. In the last and more exploratory part of the book, the mechanism for emergent gravity arising within Matrix theory is elaborated in some detail. This leads directly to gravity in $3+1$ dimensions, and provides an alternative to the more conventional approach to string theory.

The term "quantum geometry" is used here to indicate certain noncommutative mathematical structures replacing classical spacetime, analogous to the quantum mechanical description of phase space in terms of operators. In a mathematical context, this is often denoted as "noncommutative geometry," which is a priori independent of possible $\hbar$ corrections to the geometry arising from quantum theory. However, the present book is not an introduction to noncommutative geometry. Rather, it provides a systematic development of a specific physics-oriented approach to this topic, starting with the underlying mathematical structures and culminating in a specific model known as "Matrix theory." This physical theory is defined in terms of distinguished maximally supersymmetric matrix models known as Ishibashi–Kawai–Kitazawa–Tsuchiya (IKKT) and Banks–Fischer–Shenker–Susskind (BFSS) models, which are related to string theory. The main emphasis is on the IKKT model, which is best suited to exhibit a novel mechanism for gravity on quantum spacetime. The reason for this particular focus is simple: among all Yang–Mills-type matrix models, it is the unique one that is free from UV/IR pathologies upon quantization, leading to sufficiently local low-energy physics. The relation to other approaches is briefly discussed in the given context, and some supplementary topics and steps are delegated to exercises.

The book starts with a thorough discussion of the mathematical structures and tools to describe geometry through matrices, with emphasis on explicit examples known as fuzzy spaces. These are quantized symplectic spaces with finitely many degrees of freedom per unit volume. These results are then applied to physical models analogous to quantum field theory, leading to a discussion of noncommutative field theory as relevant in the context of matrix models. In particular, the novel nonlocal and stringy features are exhibited, which are often overlooked. Gauge theories are defined in terms of matrix models, which can alternatively be viewed as models for dynamical quantum geometry. Reconciling these points of view leads to a specific picture of emergent gravity, which as a classical theory differs significantly from general relativity. The appropriate geometrical structures are discussed in detail, including a particularly interesting type of covariant quantum spaces. In the last chapters, the quantization of these matrix models in terms of $\hbar$ is discussed. The explicit geometrical form of the one-loop effective action is obtained using the previously developed tools, exhibiting the Einstein–Hilbert action as a quantum effect. This suggests that gravity may be understood as a quantum effect on quantum spacetime in the framework of matrix models.

The book contains both well-known material as well as unpublished results, notably in Part III. Some of the more specialized or technical sections are marked with an asterisk and may be omitted at first reading. Parts of this book are suitable for lecture courses, notably Chapter 2 on quantization of symplectic manifolds, possibly supplemented with parts of Chapter 3 and some of the physics-related topics in the later chapters. This was tested in a two-semester course at the University of Vienna in 2021/2022, leading to numerous improvements due to attentive students, to whom I would like to express my gratitude.

I would also like to thank Emmanuele Battista, Jun Nishimura, Jurai Tekel, and Tung Tran who helped to improve the book in many ways, as well as Masanori Hanada for collaboration on Sections 5.5 and 13.3. Special thanks go to Vince Higgs from Cambridge University Press for his support during the publishing process. Finally, this book could not have been written without the help and inspiration of many colleagues and teachers during my attempts to understand fundamental physics, including Nima Arkani-Hamed, Chong-Sun Chu, Stefan Fredenhagen, Harald Grosse, Pei-Ming Ho, Hikaru Kawai, John Madore, Bogdan Morariu, Peter Schupp, Julius Wess, George Zoupanos, and Bruno Zumino, among many others.

The trouble with spacetime

Nature is governed by quantum mechanics. This was the most profound insight of twentieth-century physics, which has been established by overwhelming experimental evidence, but it was also driven by theoretical inconsistencies of classical physics. Such inconsistencies arise e.g. in the description of atoms coupled to electromagnetic radiation, which is classically unstable. This problem is cured in quantum mechanics because phase space is quantized. Indeed, all phenomena accessible to terrestrial experiments appear to be consistently described by the standard model of elementary particle physics, which is a renormalizable quantum field theory.

However, our efforts to incorporate gravity into a comprehensive quantum theory are plagued by profound difficulties. The currently accepted description of gravity is provided by general relativity (GR), which is a classical theory describing spacetime as a manifold, equipped with a dynamical metric with Lorentzian signature. This framework provides the stage for matter and fields, and hence for all known physics. General relativity also suffers from inconsistencies, such as the singularities in the center of black holes. However, GR does not seem to allow a straightforward quantization.

A simple way to see the problem is as follows. The standard model of elementary particles and interactions is governed by dimensionless coupling constants. This means that the coupling constants are the same at any distances (up to quantum corrections) and remain small. In more technical terms, these theories are called renormalizable, which means that they can be defined consistently as quantum theories at any length scale. However, gravity is governed by the dimensionful Newton constant $G_N \sim L_{Pl}^2$, where $L_{Pl} \sim 10^{-33}$ cm is the Planck length. This means that gravity becomes strongly coupled at short distances, where the quantum structure of matter and fields is significant. Since spacetime is determined by matter through the Einstein equations, that quantum structure must be strongly imprinted to spacetime at short scales. Indeed by naively applying the Einstein equations,[1] the quantum fluctuations of any field confined in a volume L_{Pl}^3 would entail strong curvature fluctuations with associated Schwarzschild radius $r_S \sim L_{Pl}$, so that the concept of a classical spacetime becomes meaningless at that scale.

In view of these issues, it seems unreasonable to insist that spacetime remains classical at all scales. It is more plausible that spacetime, matter, and fields should have a unified quantum description in a fundamental theory, possibly based on different degrees of freedom. This leads to the idea of quantizing spacetime and geometry, which goes back to the early days of quantum field theory[2] and is a recurring theme in various approaches to quantum gravity including string theory. However, the proper formulation of such a thoroughly "quantum" theory of gravity is far from obvious. In view of the aforementioned difficulties, it seems pointless trying to adapt some of the standard formulations of GR into a generalized framework. Rather, there should be a special, simple, and thoroughly well-defined definition of such an underlying theory, even if it looks unusual to the traditional eye.

Such a situation is not new in physics: the theory of strong interactions has an effective low-energy description via chiral perturbation theory, which is not renormalizable, and does not make sense as a quantum theory beyond a certain scale. The correct fundamental description is given by quantum chromodynamics, which is formulated in terms of totally different degrees of freedoms (called gluons), and makes sense as a quantum theory at any scale. It would be a bad idea to try to quantize the low-energy effective theory beyond its low-energy regime of applicability. A more mundane example to illustrate the point is given by the Navier–Stokes equations, which provide a perfectly adequate description of classical hydrodynamics above the molecular scale. We know that hydrodynamics is part of a comprehensive quantum theory, which is well understood, but it would be utter nonsense to try to quantize the Navier–Stokes equations directly beyond the molecular scale, even with the most sophisticated mathematical methods. Since the Planck scale plays

[1] There are many variants of this argument, which is more or less folklore. For a more mathematically refined discussion, see [62].
[2] Perhaps the first specific such proposal was published by Snyder [172] in 1947.

a fundamental role in gravity, it is plausible that GR provides its effective description only up to L_{Pl}, while the fundamental formulation of the theory may take a very different form.

These considerations suggest that spacetime should be described as *a dynamical physical system with intrinsic quantum structure*, treated at the same footing as the fields that live on it. Gravity should then arise through a universal metric, which governs the emergent physics. We will see that this idea can be realized through simple matrix models, where spacetime arises as solution, and physical fields arise as fluctuations of the spacetime structure. In other words, spacetime along with physical fields will *emerge* from the basic matrix degrees of freedom. If successful, this would take the idea of unification in physics one step further, as the new description is simpler than the previous one(s). There is in fact one preferred model within the framework of matrix models, known as IKKT or IIB model, which turns out to be closely related to (super)string theory.

String theory is a profound approach to reconcile gravity with quantum mechanics, which tackles the problem at its roots by giving up the concept of localized particles and fields in classical spacetime. However, there is no clear-cut comprehensive definition, and its standard world-sheet formulation leads to a number of issues. Consistency of the quantum theory requires target space to be 10-dimensional, in obvious clash with observation. This problem is typically addressed by postulating that 6 of these 10 (or rather $9 + 1$) spacetime dimensions should be curled up, or "compactified," leading effectively to a $3 + 1$ dimensional spacetime. This is a fruitful idea, which led to a wealth of new insights into the resulting quantum field theories. However, no convincing mechanism has been found which could select a realistic spacetime among the zillions of different possible compactifications. The resulting vast collection of possible worlds is dubbed the "landscape."

Avoiding this landscape problem is one motivation for the approach in this book. The matrix model(s) under consideration are closely related to string theory and thus inherit much of its magic, yet they lead to a different mechanism for $3 + 1$ dimensional spacetime and emergent (quantum) gravity, without requiring target space compactification and thereby avoiding the landscape problem. The foundations of that theory are exhibited in this book, but its detailed physical properties and its physical viability remain to be understood.

Quantum geometry and Matrix theory

This book provides an introduction to a specific approach toward a quantum theory of spacetime, matter, and gravity based on matrix models. The framework of matrix models is extremely simple, yet it provides all ingredients for a fundamental physical theory. This leads to a nonstandard formulation of field theory and gravity where the required mathematical structures emerge naturally, rather than being imposed by hand. The aim of this book is to provide the appropriate tools to understand the physical significance of these models.

All fundamental interactions are described by some type of gauge theory. Gauge theories provide the only known consistent description of quantum fields with spin. Spin 1 gauge fields include photons, gluons, and W- and Z-bosons and are described by Yang–Mills (or Maxwell) gauge fields $A_\mu(x)$. The requirement of unitarity and the absence of ghosts (i.e. negative-norm states) in quantum theory entails certain constraints for these fields, and a consistent description is achieved through gauge theories, which allow to consistently get rid of unphysical components. The same applies to gravitons in GR, which is governed by a different type of gauge theory.

There have been many attempts to formulate gauge theories on various quantum spaces, which typically encounter all sorts of issues. Rather than attempting to discuss and compare different approaches, we will focus on one approach based on matrix models. There are many reasons for this choice, but the first and foremost is simplicity; after all, a fundamental theory ought to be simple. Despite their simplicity, the models are rich enough to accommodate the basic structures required in physics. Most interestingly, they provide a novel mechanism for (quantum) gravity, which may overcome the problems in the traditional approaches.

One important message of this book is that matrix models preserve much of the power and magic of string theory. In fact, the preferred models under consideration were proposed as a constructive way to define string theory, or at least some sector of it. On the other hand, they also provide a novel mechanism for gravity on certain $3 + 1$ dimensional quantized "brane" solutions or backgrounds, which play the role of spacetime. No compactification of target space is required, thereby avoiding the landscape problem. These backgrounds are $3 + 1$ dimensional quantum geometries, and the model is expected to provide an intrinsic mechanism to choose a preferred one. Although that selection mechanism is not yet understood, matrix models certainly provide a clear-cut framework and definition, and they can even be simulated on a computer.

We will see that within the class of matrix models under consideration, there is one model that is special and preferred, and the alluded mechanism only works in this unique model. This model is known as the IKKT or IIB matrix model [102], which was first proposed in the context of string theory. We will understand the reasons for its unique standing on independent grounds, and develop a framework to understand the resulting physics. This model and its resulting physics will be denoted as *Matrix theory* in this book, even though that name is often associated with a related model of matrix quantum mechanics known as the BFSS model [21, 57]. These two models are in fact closely related and should be considered as siblings, although the proposed mechanism for emergent spacetime and gravity is better understood within the IKKT model. In any case, these models are sufficiently remarkable and rich to warrant serious efforts to understand their physics.