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978-1-009-41814-0 — Linear Algebra for Data Science, Machine Learning, and Signal Processing
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1 Getting Started

1.1 Introduction

This chapter provides introductory material, including visual examples in Section 1.2, that help motivate the rest of the book. Section 1.3 explains the book formatting and Section 1.4 previews the notation. Section 1.5 provides pointers for getting started with JULIA. Section 1.6 briefly reviews fields and vector spaces.

1.2 Example Applications

This section gives a preview of the kinds of applications that are developed in this book; these are just a few of countless applications of the material.

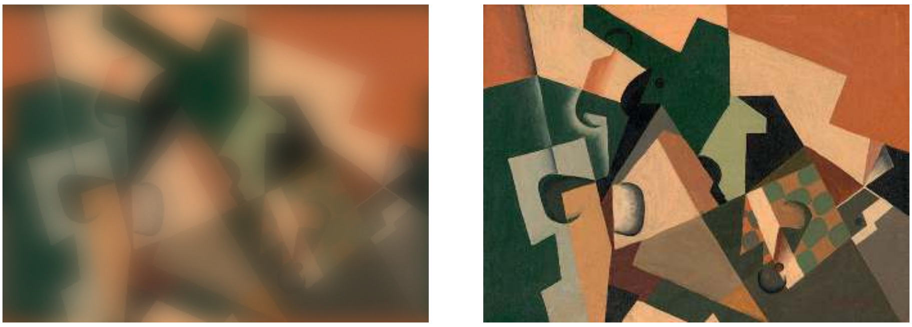
1.2.1 Signal Processing Example: Image Deblurring

One visually compelling example application of the methods discussed in this book is image deblurring, where the goal is to recover a sharp image $\hat{\boldsymbol{x}}$ from a blurry image $\boldsymbol{y}$. This problem has numerous applications, including in science, such as correcting for optical imperfections in the Hubble space telescope [13], and in forensics, as seen in television crime dramas using low-quality security cameras. Figure 1.1 shows an example of an out-of-focus image of a painting (a) and the recovered sharp image (b).

Each of the images in Fig. 1.1 is an array of 1271×948 (width $\times$ height) pixels, where each pixel has a color represented by red–green–blue (RGB) values called color channels. Here, instead of thinking of each color channel as a 1271×948 matrix, we "vectorize" all of the red pixels into a vector of length $1271 \cdot 948$; see (2.18). Likewise for the green and blue channels. For the example shown here, we process each channel separately.

The first step in any such data-processing problem is to formulate a model that relates the known data to the unknown or latent parameters. The image $\boldsymbol{y}$ in Fig. 1.1 has 1271×948 pixels, so a typical model is

$$\underbrace{\boldsymbol{y}}_{\mathbb{R}^{1271 \cdot 948}} = \boldsymbol{A} \underbrace{\boldsymbol{x}}_{\mathbb{R}^{1271 \cdot 948}} + \boldsymbol{\varepsilon}, \tag{1.1}$$



(a) (b)

www.nga.gov/collection/art-object-page.166491.html

Figure 1.1 *Glass and Checkerboard* by Juan Gris before (a) and after (b) deblurring.

where A is a matrix of size $1\,204\,908 \times 1\,204\,908$ describing the optical blur and ε represents sensor noise. Under some simplifications, a solution for recovering the image x is

$$\hat{x} = \underset{x \in \mathbb{R}^{1271 \cdot 948}}{\arg\min} \; \|Ax - y\|_2^2 = A^+ y = V \Sigma^+ U' y. \tag{1.2}$$

This formulation and solution involves the majority of the chapters in this book!

- The problem and solution is expressed using matrices and vectors (Chapter 2).
- The solution is expressed using a singular value decomposition (Chapter 3).
- The formulation is a least-squares problem and the solution involves a matrix pseudo inverse (Chapter 5).
- The formulation here uses the Euclidean norm for vectors (Chapter 6).
- The "arg min" represents an optimization problem (Section 4.9, Chapter 9).
- One way to solve this problem efficiently is to use fast Fourier transform (FFT) operations because the matrix A here has a special structure called block circulant with circulant blocks (Section 8.3).

State-of-the-art methods for image restoration use even more advanced methods such as 1-norm regularizers (Chapter 6) and deep neural networks (Chapter 11). A foundation in the matrix methods of this book is helpful for reading the literature on those methods.

1.2.2 Computer Vision Applications

Another imaging example, shown in Section 7.2.11, is photometric stereo; see Fig. 7.6 and Demo 7.2. Section 10.5.1 illustrates foreground/background separation in video; see Fig. 10.4 and Demo 10.3.

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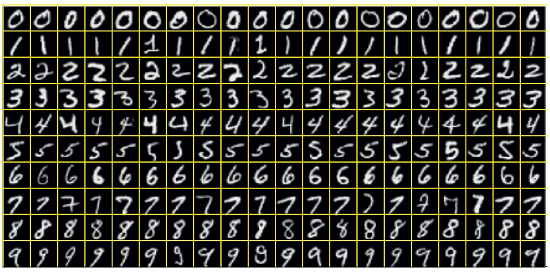

Figure 1.2 Labeled training data for handwitten digit recognition from the MNIST dataset [14].

1.2.3 Machine Learning Example: Handwritten Digit Recognition

One machine learning application that is a recurring theme in this book is handwritten digit recognition. Given labeled training data, like the images of handwritten digits shown in Fig. 1.2, how does one design a classifier that can automatically predict the digit seen in a test image? See Sections 4.8.3, 5.9.4, and 9.5.

1.3 Formatting

Here is an overview of how material is formatted in this book.

- JULIA `code snippets are shown like this.`
- A "dangerous bend" symbol in the margin warns of tricky material.
- References to homework problems look like: Problem 2.1.
- A double diamond symbol in the margin is "experts only" material that is included for reference but is generally not needed subsequently in the book.

Definition Key definitions are shown like this.

Fact 1.1 Important facts are shown like this.

Particularly important topics are shown like this.

Demo 1.1 provides a JULIA overview. Demos corresponding to online examples containing both code and results are shown like this and are available at the URL shown in the preface.

Example 1.1 Examples are shown like this.

</> 1.1

```
# Julia code files are shown like this.
@assert exp(1im * π) ≈ -1
```

Such JULIA code files are also available on the code website, arranged by chapter with the numbers shown in the margin.

Explore 1.1 In-line exploration problems are shown like this, with answers at the end of the chapter. The number is a hyperlink in the e-version; a click on it should cause the viewer to jump to the solution.

Q1.1 Questions designed for interactive classroom use are formatted like this.
(Answers are available to instructors through the publisher.)
A: True B: False

1.4 Notation Preview

Here is a preview of notation that is used throughout. Most of these are standard in math and linear algebra texts. For the less familiar ones, the definitions are given later in the text.

Symbol	What	Where
Numbers		
$\mathbb{N}$	natural numbers $1, 2, \ldots$	
$\mathbb{Z}$	integers	
$\mathbb{Q}$	rational numbers	p. 8
$\mathbb{R}$	real numbers	
$\mathbb{R}_+$	nonnegative real numbers	p. 135
$\mathbb{C}$	complex numbers	
$\mathbb{F}$	field of scalars, typically $\mathbb{R}$ or $\mathbb{C}$	p. 8
$\imath$	$\sqrt{-1}$ imaginary unit	p. 21
z^*	complex conjugate of $z \in \mathbb{C}$	p. 21
$\lvert z \rvert$	complex modulus or absolute value	p. 21
$\angle z$	angle of complex number	p. 21
$\mathrm{sign}(z)$	sign of (possibly complex) number z: $\mathrm{sign}\left(\lvert z\rvert\, \mathrm{e}^{\imath \angle z}\right) = \mathrm{e}^{\imath \angle z}$	p. 82
Logic		
$\Longrightarrow$	logical implication	
$\Longleftrightarrow$	if and only if	
$\forall$	for all	
$\exists$	exists	

Symbol	What	Where
Sets		
$\{a, b, \ldots\}$	set (unordered collection)	
$(a, b, \ldots)$	tuple (ordered collection)	
$\emptyset$	empty set	p. 99
$\in$	is a member of (a set)	
$\subset$	subset of	
$\cap$	set intersection	
$\cup$	set union	
$\mathcal{A}, \mathcal{B}, \ldots$	upper case caligraphic denotes sets	
$\mathcal{A} \setminus \mathcal{B}$	set difference (relative complement)	p. 53
Calculus and functions		
$\mapsto$	maps to	p. 11
$\dot{f}$	derivative of function $f : \mathbb{R} \mapsto \mathbb{R}$	p. 148
∇f	gradient of function $f : \mathbb{F}^N \mapsto \mathbb{R}$	p. 148
$g \circ f$	function composition	p. 131
$\mathbb{I}_{\{\cdot\}}$	indicator function (1 or 0)	p. 16
$\cos.(\boldsymbol{x})$	broadcast operation (elementwise)	p. 36
$[x]_+$	$\max(x, 0)$	p. 259
Matrices and linear algebra		
$\mathcal{S}, \mathcal{V}$	subspace, vector space	p. 96
$\mathcal{S}^\perp$	subspace orthogonal complement	p. 107
$\boldsymbol{P}^\perp$	project onto orthogonal complement	p. 133
$\subseteq$	subspace of	p. 96
$\oplus$	subspace direct sum	p. 106
$\otimes$	Kronecker product	p. 34
$\odot$	Hadamard product (elementwise)	p. 34
$\boldsymbol{A}, \ldots, \boldsymbol{Z}$	matrices: bold upper-case letters	
$\boldsymbol{I}$	identity matrix	p. 18
$\boldsymbol{a}, \ldots, \boldsymbol{z}$	vectors: bold lower-case letters	
x_n	nth element of vector $\boldsymbol{x}$	
a_{mn}	element in row m, column n of matrix $\boldsymbol{A}$	
$\boldsymbol{A}^\top$	transpose of matrix $\boldsymbol{A}$	p. 20
$\boldsymbol{A}'$	Hermitian transpose of matrix $\boldsymbol{A}$	p. 20
$\boldsymbol{A}^+$	Moore–Penrose pseudoinverse of matrix $\boldsymbol{A}$	p. 155
λ_k	eigenvalues of a square matrix	p. 50
σ_k	singular values of a matrix	p. 74
$\langle \boldsymbol{x}, \boldsymbol{y} \rangle$	inner product	p. 203
$\boldsymbol{x} \perp \boldsymbol{y}$	orthogonality	p. 42
$\|\boldsymbol{x}\|$	vector norm	p. 42
$\|\|\|\boldsymbol{A}\|\|\|$	submultiplicative matrix norm	p. 206

Symbol	What	Where
$\mathbf{0}$	vector (or matrix) of all zeros	p. 9
$\mathbf{1}$	vector of all ones	p. 36
$\boldsymbol{A} \succeq \mathbf{0}$	$\boldsymbol{A}$ is positive semidefinite	p. 88
$\boldsymbol{A} \succ \mathbf{0}$	$\boldsymbol{A}$ is positive definite	p. 88
$\boldsymbol{A} \succeq \boldsymbol{B}$	$\boldsymbol{A} - \boldsymbol{B}$ is positive semidefinite	p. 341
$\boldsymbol{A} \succ \boldsymbol{B}$	$\boldsymbol{A} - \boldsymbol{B}$ is positive definite	p. 341
$\mathsf{Diag}\{\boldsymbol{x}\}$	diagonal matrix from elements of vector $\boldsymbol{x}$	p. 18
$\mathcal{R}(\boldsymbol{A})$	range of matrix $\boldsymbol{A}$	p. 109
$\mathcal{N}(\boldsymbol{A})$	nullspace of matrix $\boldsymbol{A}$	p. 116
$\kappa(\boldsymbol{A})$	condition number of matrix $\boldsymbol{A}$	p. 164
$\rho(\boldsymbol{A})$	spectral radius of matrix $\boldsymbol{A}$	p. 217
$\boldsymbol{e}_i$	ith unit vector (standard basis)	p. 24
Vectors and matrices		
$\mathbb{R}^N$	N-tuples of real numbers	p. 9
$\mathbb{C}^N$	N-tuples of complex numbers	p. 9
$\mathbb{F}^N$	either $\mathbb{R}^N$ or $\mathbb{C}^N$	
$\mathbb{R}^{M \times N}$	real $M \times N$ matrices	
$\mathbb{C}^{M \times N}$	complex $M \times N$ matrices	
$\mathbb{F}^{M \times N}$	either $\mathbb{R}^{M \times N}$ or $\mathbb{C}^{M \times N}$	

1.4.1 What the $\mathbb{F}$ Means

The notation $\mathbb{F}$ for a scalar field used here will typically be $\mathbb{R}$ or $\mathbb{C}$. We will use this symbol frequently to discuss properties that hold for both real and complex cases. Because $\mathbb{R}^{M \times N} \subset \mathbb{C}^{M \times N}$, whenever you see the symbol $\mathbb{F}^{M \times N}$ you can just think of it as $\mathbb{C}^{M \times N}$, but it means the properties being discussed also hold for $\mathbb{R}^{M \times N}$.

To explain the convenience of using the $\mathbb{F}$ notation, consider the following logical implication statements about vector addition, that is, $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$, all of which are true:

(i) $\boldsymbol{x} \in \mathbb{R}^N, \boldsymbol{y} \in \mathbb{R}^N \Longrightarrow \boldsymbol{z} \in \mathbb{R}^N$.
(ii) $\boldsymbol{x} \in \mathbb{C}^N, \boldsymbol{y} \in \mathbb{C}^N \Longrightarrow \boldsymbol{z} \in \mathbb{C}^N$.
(iii) $\boldsymbol{x} \in \mathbb{R}^N, \boldsymbol{y} \in \mathbb{C}^N \Longrightarrow \boldsymbol{z} \in \mathbb{C}^N$.
(iv) $\boldsymbol{x} \in \mathbb{C}^N, \boldsymbol{y} \in \mathbb{R}^N \Longrightarrow \boldsymbol{z} \in \mathbb{C}^N$.
(v) $\boldsymbol{x} \in \mathbb{R}^N, \boldsymbol{y} \in \mathbb{R}^N \Longrightarrow \boldsymbol{z} \in \mathbb{C}^N$.

Because $\mathbb{R} \subset \mathbb{C}$, statement (ii) implies (iii), (iv), and (v). However, (ii) does *not* imply (i)! Of course (i) is true, but simply stating or proving (ii) by itself is insufficient to conclude that (i) is true. So to thoroughly describe vector addition one would need to prove (or state as fact) both (i) and (ii) separately. Instead, this book uses statements like this:

$$\boldsymbol{x} \in \mathbb{F}^N, \boldsymbol{y} \in \mathbb{F}^N \Longrightarrow \boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y} \in \mathbb{F}^N,$$

which is essentially shorthand for stating both (i) and (ii) above. Proving a statement like that may require separately proving (i) and (ii).

1.5 Julia

All code examples in this book use the open-source JULIA programming language, which is designed for numerical computing [1]. It is interactive (like Python and MATLAB), yet has fast execution because it is compiled. The examples in this book were developed using JULIA 1.9, but should work with JULIA 1.6 or higher. We recommend using the latest 1.x version because JULIA speed tends to improve each version while retaining backward compatibility. To get started with JULIA, install the free application from https://julialang.org. For additional tips about editors, tutorials, and so forth, see https://jefffessler.github.io/book-la-demo.

Table 1.1 briefly compares three interactive languages. The JULIA column assumes you have typed `using LinearAlgebra` to use the `dot` function.

1.6 Fields, Vector Spaces, Linear Maps

There are multiple different meanings of the term vector in the literature.

- In MATLAB, a vector is simply a column of a two-dimensional (2D) matrix. In other words, a MATLAB (column) vector is an $N \times 1$ array of numbers, that is, a 2D array where the second dimension is 1.

Table 1.1. Syntax comparison.

Operation	MATLAB	JULIA	Python `import numpy as np`
Dot product	`dot(x,y)`	`dot(x,y)`	`np.dot(x,y)`
Matrix multiplication	`A * B`	`A * B`	`A @ B`
Elementwise	`A .* B`	`A .* B`	`A * B`
Scaling	`3 * A`	`3A or 3*A`	`3 * A`
Matrix power	`A^2`	`A^2`	`np.linalg.matrix_power(A,2)`
Elementwise	`A.^2`	`A.^2`	`A**2`
Inverse	`inv(A)`	`inv(A)`	`np.linalg.inv(A)`
Inverse	`A^(-1)`	`A^(-1)`	`np.linalg.inv(A)`
Indexing	`A(i,j)`	`A[i,j]`	`A[i-1,j-1]`
Range	`1:9`	`1:9`	`np.arange(1,10,1)`
Range	`linspace(0,4,9)`	`range(0,4,9)`	`np.arange(0,4.01,0.5)`
Strings	`'text'` or `"text"`	`"text"`	`'text'` or `"text"`
Slicing	`v(1:end)`	`v[1:end]`	`v[0:]`
Inline function	`f = @(x,y) x+y`	`f(x,y) = x+y`	`f = lambda x,y : x+y`
Increment	`A = A + B`	`A += B`	`A += B`
Hermitian transpose	`A'`	`A'`	`A.conj().T`
$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$	`[1 2; 3 4]`	`[1 2; 3 4]`	`np.array([[1, 2], [3, 4]])`

- In JULIA, a `Vector` is a 1D array of N values, typically numbers. This convention is more consistent with numerical linear algebra. More precisely, a variable of type `Vector{<:Number}` is a 1D array of numbers. The JULIA `Vector` type is general enough to hold any elements.
- In general mathematics, for example, linear algebra and functional analysis, a vector belongs to a vector space.

Although this book mostly uses the numerical methods perspective, readers who want a thorough understanding should also be familiar with the more general notion of a vector space.

Demo 1.2 provides more illustrations of how vectors work in JULIA.

This section reviews vector spaces and linear operators defined on vector spaces. Although the definitions in this section are quite general and thus might appear somewhat abstract, the ideas are important even for the topics of an undergraduate signals and systems course. For example, analog systems like passive RLC networks are linear systems that are represented mathematically by linear maps from one (infinite-dimensional) vector space to another.

The definition of a vector space uses the concept of a field of scalars, so we first review that.

1.6.1 Field of Scalars

A field or field of scalars $\mathbb{F}$ is a collection of elements $\alpha, \beta, \gamma, \ldots$ along with an "addition" and a "multiplication" operator [15].

For every pair of scalars α, β in $\mathbb{F}$, there must correspond a scalar $\alpha + \beta$ in $\mathbb{F}$, called the sum of α and β, such that:

- Addition is commutative: $\alpha + \beta = \beta + \alpha$.
- Addition is associative: $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.
- There exists a unique element $0 \in \mathbb{F}$, called zero, for which $\alpha + 0 = \alpha, \forall \alpha \in \mathbb{F}$.
- For every $\alpha \in \mathbb{F}$, there corresponds a unique scalar $(-\alpha) \in \mathbb{F}$ for which $\alpha + (-\alpha) = 0$.

For every pair of scalars α, β in $\mathbb{F}$, there must correspond a scalar $\alpha\beta$ in $\mathbb{F}$, called the product of α and β, such that:

- Multiplication is commutative: $\alpha\beta = \beta\alpha$.
- Multiplication is associative: $\alpha(\beta\gamma) = (\alpha\beta)\gamma$.
- Multiplication distributes over addition: $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.
- There exists a unique element $1 \in \mathbb{F}$, called one, or unity, or the identity element, for which $1\alpha = \alpha, \forall \alpha \in \mathbb{F}$.
- For every nonzero $\alpha \in \mathbb{F}$, there corresponds a unique scalar $\alpha^{-1} \in \mathbb{F}$, called the inverse of α, for which $\alpha\alpha^{-1} = 1$.

Example 1.2 The set $\mathbb{Q}$ of rational numbers is a field (with the usual definitions of addition and multiplication).

The primary fields needed in this book are the field of real numbers $\mathbb{R}$ and the field of complex numbers $\mathbb{C}$.

1.6.2 Vector Spaces

A vector space or linear space consists of:

- a field $\mathbb{F}$ of scalars;
- a set $\mathcal{V}$ of entities called vectors;
- an operation called vector addition that associates a sum $\boldsymbol{x} + \boldsymbol{y} \in \mathcal{V}$ with each pair of vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$ such that:
  - addition is commutative: $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$;
  - addition is associative: $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$;
  - there exists a unique element $\boldsymbol{0} \in \mathcal{V}$, called the zero vector, for which $\boldsymbol{x} + \boldsymbol{0} = \boldsymbol{x}$, $\forall \boldsymbol{x} \in \mathcal{V}$;
  - For every $\boldsymbol{x} \in \mathcal{V}$, there corresponds a unique vector $(-\boldsymbol{x}) \in \mathcal{V}$ for which $\boldsymbol{x} + (-\boldsymbol{x}) = \boldsymbol{0}$;
- an operation called multiplication by a scalar that associates with each scalar $\alpha \in \mathbb{F}$ and vector $\boldsymbol{x} \in \mathcal{V}$ a vector $\alpha\boldsymbol{x} \in \mathcal{V}$, called the product of α and $\boldsymbol{x}$, with the following properties:
  - associative: $\alpha(\beta\boldsymbol{x}) = (\alpha\beta)\boldsymbol{x}$;
  - distributive $\alpha(\boldsymbol{x} + \boldsymbol{y}) = \alpha\boldsymbol{x} + \alpha\boldsymbol{y}$;
  - distributive $(\alpha + \beta)\boldsymbol{x} = \alpha\boldsymbol{x} + \beta\boldsymbol{x}$;
  - if 1 is the identity element of $\mathbb{F}$, then $1\boldsymbol{x} = \boldsymbol{x}$, $\forall \boldsymbol{x} \in \mathcal{V}$.

(We do not require other operations to be defined, such as multiplying two vectors or adding a vector and a scalar.)

We describe such a space as an $\mathbb{F}$-vector space or a vector space over $\mathbb{F}$.

Example 1.3 An example of a vector space is the real coordinate space of dimension n, or Euclidean n-dimensional space, or n-tuple space: $\mathcal{V} = \mathbb{R}^n$. If $\boldsymbol{x} \in \mathcal{V}$, then $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ where $x_i \in \mathbb{R}$. The field of scalars is $\mathbb{F} = \mathbb{R}$. Of course, $\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$, and $\alpha\boldsymbol{x} = (\alpha x_1, \ldots, \alpha x_n)$.

This space is very closely related to the notion of a vector being a column of a matrix. They are so close that much of the literature does not distinguish them (nor does MATLAB). However, to be rigorous, $\mathbb{R}^n$ is not the same as a column of a matrix because strictly speaking there is no inherent definition of the transpose of a vector in $\mathbb{R}^n$, whereas transpose is well defined for any matrix including an $n \times 1$ matrix. See Section 2.3.2.

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Vectors in $\mathbb{F}^N$ are typically thought of as column vectors here. When needed, row vectors are written as $\boldsymbol{x}^\top$ or $\boldsymbol{x}'$, where $\boldsymbol{x} \in \mathbb{R}^N$ or $\boldsymbol{x} \in \mathbb{C}^N$.

Example 1.4 The complex coordinate space of dimension n or complex Euclidean n-dimensional space: $\mathcal{V} = \mathbb{C}^n$. If $\boldsymbol{x} \in \mathcal{V}$, then $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ where $x_i \in \mathbb{C}$. The field of scalars is $\mathbb{F} = \mathbb{C}$, $\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, \ldots, x_n + y_n)$, and $\alpha \boldsymbol{x} = (\alpha x_1, \ldots, \alpha x_n)$.

Example 1.5 The set of functions $f : \mathbb{R} \mapsto \mathbb{C}$ that are continuous. Addition and scalar multiplication are defined in the natural way. This is a vector space because linear combinations of continuous functions are continuous.

Example 1.6 The set of functions on the plane $\mathbb{R}^2$ that are zero outside of the unit square.

Example 1.7 The set of solutions to a homogeneous linear system of equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$.

Example 1.8 The set $\mathcal{V}$ of diagonal matrices whose diagonal elements are rational numbers, over the field $\mathbb{Q}$.

- If $\boldsymbol{A}$ and $\boldsymbol{B}$ are both in $\mathcal{V}$, then their sum $\boldsymbol{A} + \boldsymbol{B}$ is also a diagonal matrix whose elements are rational numbers.
- If $\boldsymbol{A} \in \mathcal{V}$ and $\alpha \in \mathbb{Q}$ is a rational number, then $\alpha \boldsymbol{A}$ is again a diagonal matrix whose elements are rational numbers.

The other properties are easily verified.

In general, the vectors in a vector space need not consist of elements that are taken from the field $\mathbb{F}$.

Example 1.9 The set $\mathcal{V}$ of $M \times N$ matrices whose elements are complex numbers, over the field $\mathbb{R}$.

- If $\boldsymbol{A}$ and $\boldsymbol{B}$ are both in $\mathcal{V}$, then their sum $\boldsymbol{A} + \boldsymbol{B}$ is also an $M \times N$ matrix whose elements are complex numbers.
- If $\boldsymbol{A} \in \mathcal{V}$ and $\alpha \in \mathbb{R}$ is a real number, then $\alpha \boldsymbol{A}$ is again an $M \times N$ matrix whose elements are complex numbers.

The other properties are easily verified.

The preceding two examples serve as a reminder that the concept of vector is quite general, and even includes matrices!