AN INTRODUCTION TO GENERAL RELATIVITY AND COSMOLOGY

Experts Plebański and Krasiński provide a thorough introduction to the tools of general relativity and relativistic cosmology. Assuming familiarity with advanced calculus, classical mechanics, electrodynamics and special relativity, the text begins with a short course on differential geometry, taking a unique top-down approach. Starting with general manifolds on which only tensors are defined, the covariant derivative and affine connection are introduced before moving on to geodesics and curvature. Only then is the metric tensor and the (pseudo)-Riemannian geometry introduced, specialising the general results to this case. The main text describes relativity as a physical theory, with applications to astrophysics and cosmology. It takes the reader beyond traditional courses on relativity through in-depth descriptions of inhomogeneous cosmological models and the Kerr metric. Emphasis is given to complete and clear derivations of the results, enabling readers to access research articles published in relativity journals.

JERZY PLEBAŃSKI (1928–2005) was a Polish theoretical physicist best known for his extensive research into general relativity, nonlinear electrodynamics and mathematical physics. He split his time between Warsaw, Poland and Mexico, his permanent residence from the mid-1970s onwards. He is remembered, among other things, for defining the algebraic classification of the tensor of matter, for finding new solutions of the Einstein equations (for example, the Plebański–Demiański metric), formulation of the heavenly equations and the effective field theory relating GR and supergravity, known as Plebański action. The first part of the book is developed from Plebański's lecture notes.

ANDRZEJ KRASIŃSKI is Emeritus Professor at the Nicolaus Copernicus Astronomical Center, Polish Academy of Sciences in Warsaw. He served for many years on the Editorial Board of the journal *General Relativity and Gravitation* and has acted as Poland's elected representative on the International Committee of the International Society for General Relativity and Gravitation. He is author of *Inhomogeneous Cosmological Models* (Cambridge, 1997), co-author of *Structures in the Universe by Exact Methods* (Cambridge, 2009) and co-editor of *Golden Oldies in General Relativity* (Springer, 2013). His research record includes many papers on the interpretation of inhomogeneous cosmological models. He is also a co-author of the computer program, Ortocartan, for algebraic calculations in general relativity.

AN INTRODUCTION TO GENERAL RELATIVITY AND COSMOLOGY

JERZY PLEBAŃSKI
National Polytechnic Institute of Mexico

ANDRZEJ KRASIŃSKI
Nicolaus Copernicus Astronomical Center, Polish Academy of Sciences

CAMBRIDGE UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

Shaftesbury Road, Cambridge CB2 8EA, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre,
New Delhi – 110025, India

103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of Cambridge University Press & Assessment,
a department of the University of Cambridge.

We share the University's mission to contribute to society through the pursuit of
education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781009415620

DOI: 10.1017/9781009415651

First published 2006
Second edition published 2024

A catalogue record for this publication is available from the British Library

A Cataloging-in-Publication data record for this book is available from the Library of Congress

ISBN 978-1-009-41562-0 Hardback

Contents

The scope of this text

General relativity is the currently accepted theory of gravitation. Under this heading one could include a huge amount of material. For the needs of this theory an elaborate mathematical apparatus was created. It has partly become a self-standing sub-discipline of mathematics and physics, and it keeps developing, providing input or inspiration to physical theories that are being newly created (such as gauge field theories, supergravitation and the brane-world theories). From the gravitation theory, descriptions of astronomical phenomena taking place in strong gravitational fields and in large-scale sub-volumes of the Universe are derived. This part of gravitation theory develops in connection with results of astronomical observations. For the needs of this area, another sophisticated formalism was created (the Parametrised Post-Newtonian, PPN, formalism). Finally, some tests of the gravitation theory can be carried out in laboratories, either terrestrial or orbital. These tests, their improvements and projects of further tests have led to developments in mathematical methods and in technology that are by now an almost separate branch of science – as an example, one can mention here the search for gravitational waves and the calculations of properties of the wave signals to be expected.

In this situation, no single textbook can attempt to present the whole of gravitation theory, and the present text is no exception. We made the working assumption that relativity is part of physics (this view is not universally accepted!). The purpose of this course is to present those results that are most interesting from the point of view of a physicist, and were historically the most important. We are going to lead the reader through the mathematical part of the theory by a rather short route, but in such a way that he/she does not have to take anything on our word, is able to verify every detail and, after reading the whole text, will be prepared to solve several problems by him/herself. Further help in this should be provided by the exercises in the text and the literature recommended for further reading.

The introductory part (Chapters 1–7), although assembled by J. Plebański long ago, has never been published in book form.[1] It differs from other courses on relativity in that it introduces differential geometry by a top-down method. It begins with general manifolds, on which no structures except tensors are defined, and discusses their basic properties. Then it adds the notion of the covariant derivative and affine connection,

[1] A part of that material had been semi-published as copies of typewritten notes (Plebański, 1964).

without introducing the metric yet, and again proceeds as far as possible. At that level it defines geodesics via parallel displacement and presents the properties of curvature. Only at this point it introduces the metric tensor and the (pseudo) Riemannian geometry and specialises the results derived earlier to this case. Then it proceeds to the presentation of more detailed topics, such as symmetries, the Bianchi classification and the Petrov classification.

Some of the chapters on classical relativistic topics contain material that, to the best of our knowledge, has never been published in any textbook. In particular, this applies to Chapter 8 (on symmetries) and to Chapter 16 (on cosmology with general geometry). Chapters 18–20 (on inhomogeneous cosmologies) are entirely based on original papers. Parts of Chapters 18, 19 and 20 cover the material introduced in A. K.'s monograph on inhomogeneous cosmological models (Krasiński, 1997). However, the presentation here was thoroughly rearranged, extended and brought up to date. We no longer intended to briefly mention all contributions to the subject; rather, we have placed the emphasis on complete and clear derivations of the most important results. That material has so far existed only in scattered journal papers and has been assembled into a textbook for the first time (the monograph by Krasiński, 1997, was only a concise review). Taken together, this collection of knowledge constitutes an important and interesting part of relativistic cosmology whose meaning has, unfortunately, not yet been appreciated properly by the astronomical community.

Most figures for this text, even when they look the same as the corresponding figures in the published papers, were newly generated by A. K. using the program Gnuplot, sometimes on the basis of numerical calculations programmed in Fortran 90. The only figures taken verbatim from other sources are those that illustrated the joint papers of A. K. with C. Hellaby and K. Bolejko, and of N. Ashby in Chapter 22. The latter are reproduced here with the permission of the author and of N. Dadhich as a representative of the publisher, Inter-University Centre for Astronomy and Astrophysics in Pune, India.

J. Plebański kindly agreed to be included as a co-author of this text – having done his part of the job long ago. Unfortunately, he was not able to participate in the writing up and proofreading. He died while the first edition of this book was being edited. Therefore, the second author (A. K.) is exclusively responsible for any errors.

Note for the reader. Some parts of this book may be skipped on first reading, since they are not necessary for understanding the material that follows. They are marked by asterisks. Chapters 18–20 are expected to be the highlights of this book. However, they go far beyond standard courses of relativity and may be skipped by those readers who wish to remain on the well-beaten track. Hesitating readers may read on, but can skip the sections marked by asterisks.

Andrzej Krasiński
Warsaw, September 2005 and May 2023

Preface to the second edition

The main reason why a new edition of this book was needed is that during the 17 years after the first edition several little errors have been discovered in it. Mostly, they are innocent typos that can be corrected by a careful reader (most of them were discovered by my then-student, Mr Przemysław Jacewicz, to whom I express my gratitude). However, a few required significant changes. The list of those changes is given further below.

By this opportunity, a few sections and a new chapter were added. The added sections describe the developments in understanding the cosmological implications of the Lemaître–Tolman and Szekeres models that occurred after 2006. The added Chapter 22, on relativistic effects in the Global Positioning System, is a thought-provoking demonstration that general relativity became a necessary component in today's down-to-Earth technology.

A few reviewers of the first edition complained that this book is not really an introduction to relativity and cosmology, but rather an advanced-level monograph, so its title is misleading. Still, together with the publisher, we decided to keep the 'introduction' in the title, for the following reasons:

1. No prior familiarity of the reader with general relativity and differential geometry is assumed in this book. It takes a careful reader to some height of advancement, but it begins at a very basic level.
2. The book omits several topics (see Chapter 23). For example, the reviewers of the first edition complained about the omission of perturbative methods in cosmology. Consequently, this is not a complete monograph. To become an expert in relativity the reader will have to continue his/her reading with other books. The reason for the omissions was that we intended to make our readers familiar with the mathematical and physical basics of relativity, leaving out those applications that require extended additional study. *Ergo*, our book is not an encyclopaedia of the subject, but ... well, an introduction.

My co-author and former master, Jerzy Plebański, died in 2005. But the part of this book that is directly based on his works and on my notes to his lectures is still a solid fundament of the whole.

The list of significant changes from the first edition

Several figures were redrawn to make them graphically clearer or to make the symbols in them consistent with the text.

Section 9.4 (on algebraic computing) was shortened because algebraic computing is no longer a novelty.

Section 10.5 (on invariant vector fields) was re-edited for better clarity.

Chapter 11 (on spinors) was re-edited and rearranged to improve the clarity of presentation, and extended hints were added to most exercises.

Section 12.16 (on other theories of gravitation) was re-edited.

Section 12.18 (on the weak-field approximation to general relativity) was re-edited to improve the clarity of presentation, and the information about the Gravity Probe B experiment was updated.

Sections 14.6 (on measuring the deflection of light rays) and 14.8 (on gravitational lenses) were re-edited to make them more up to date, and photographs of two gravitational lenses were added at the end of Sec. 14.8.

A part of Chapter 16 (on cosmology in general geometry) was reshuffled to achieve a more logical ordering of the exposition.

Chapter 17 (on cosmology in Robertson–Walker geometry) was substantially re-edited and updated in several places.

Chapter 18 (on cosmology in Lemaître–Tolman geometry) was re-edited in several places. In particular,

- Section 18.18 (on singularities and cosmic censorship) was re-edited and shortened.
- The old Subsection 18.20.2 (now 18.24.2) had to be partly corrected, but the conclusion remains unchanged.

The old Chapter 19 was split into two. The old Sections 19.1–19.5 became the new Chapter 19 under the title 'Relativistic cosmology IV: simple generalisations of L–T and related geometries'. By this opportunity, it was extended for results obtained after the first edition (see below). The remaining part of the old Chapter 19 became Chapter 20 with the title: 'Relativistic cosmology V: the Szekeres geometries'.

Consequently, the old Chapter 20 (on the Kerr metric) became Chapter 21.

The old Sec. 19.5.3 (now Sec. 20.1.3) was corrected at several places.

The old Subsection 19.6.3 (now Subsection 20.2.3) was completely rewritten to get rid of several errors.

The old Subsection 19.7.6 (now Subsection 20.3.7) was completely rewritten to put its terminology into agreement with the existing literature.

The old Subsection 19.7.7 (on nonexistent Szekeres wormholes) was removed.

The old Section 19.10, containing the proof that one subcase in solving the Einstein equations for the Szekeres-type metrics leads to a contradiction, was transferred to the new Chapter 24 and is now Section 24.7. It contains a large amount of calculations with a rather disappointing end result.

Chapter 21 (old number 20) was re-edited in several places to improve the clarity of presentation.

The following new material was added:

(i) A new Section 8.5 on the relation between the Killing vectors and geodesic deviation.

(ii) New Exercises 9 and 14 at the end of Chapter 8.
(iii) Several new exercises at the end of Chapter 12.
(iv) At the end of Chapter 16:
- A derivation of the formula for the Fermi–Walker transport and references to the original Fermi and Walker papers.
- A derivation of the formula for the position drift of a light source in a general geometry, following the paper by M. Korzyński and J. Kopiński (2018).

(v) In Chapter 17, the following three new sections:
- Section 17.8, introducing the ΛCDM model.
- Section 17.9, with the derivation of the distance–redshift relation in the Friedmann models with Λ.
- Section 17.10, on the redshift drift in Robertson–Walker models.

(vi) Several new exercises at the end of Chapter 17.
(vii) Section 18.12, explaining the reasons why some objects in the Universe may display blueshift rather than redshift.
(viii) Section 18.13, describing how apparent horizons in L–T models are radically different for noncentral observers.
(ix) At the end of Chapter 18 – a new section that demonstrates how the accelerating expansion of the Universe can be mimicked by inhomogeneities in mass distribution.
(x) Also at the end of Chapter 18 – an exemplary illustration of the rate of change of position of a distant galaxy on the sky caused by the drift described in the newly added section at the end of Chapter 16.
(xi) A new Section 19.3, in which it is argued that spaces of constant time in plane symmetric spacetimes are most naturally interpreted as flat tori.
(xii) At the end of the new Sec. 19.4.6 – the demonstration how a charged dust ball can be evolved through the minimal size, only to hit a shell crossing singularity after that.
(xiii) Second half of the new Section 19.5 – a discussion of evolution of the charged Ruban (1972, 1983) solution matched to Reissner–Nordström (RN).
(xiv) At the end of the new Section 19.5 – spacetime diagrams of the configuration mentioned above and of the neutral Ruban (1968, 1969) model matched to Schwarzschild.
(xv) The new Subsections 20.3.8 and 20.3.9, in which the 'absolute apparent horizon' (AAH) and the ordinary apparent horizon (AH) of the central observer in the quasi-spherical Szekeres spacetimes are compared.
(xvi) At the end of the new Chapter 20, a new Section 20.6 with short descriptions of a few papers published after the first edition of this book that contain instructive contributions to the Szekeres geometries.
(xvii) Several new exercises to Chapter 21 (old Chapter 20).
(xviii) A new Chapter 22 on the relativistic effects in the Global Positioning System.
(xix) Another new Chapter 24 that contains detailed hints on how to solve the more difficult exercises and verify complicated calculations.

Andrzej Krasiński
Warsaw, 2023

Acknowledgements

We thank Charles Hellaby for comments on the various properties of the Lemaître–Tolman models and for providing copies of his unpublished works on this subject. Some of the figures used in this text were copied from C. Hellaby's files, with his permission. We are grateful to Pankaj S. Joshi for helpful comments on cosmic censorship and singularities, and to Amos Ori for clarifying the matter of shell crossings in charged dust. The correspondence with Amos significantly contributed to clarifying several points in Section 19.4. We are also grateful to George Ellis for his very useful comments on the first draft of this book. We thank Bogdan Mielnik and Maciej Przanowski, who were of great help in the difficult communication between one of the authors residing in Poland and the other in Mexico. M. Przanowski had carefully proofread a large part of this text and caught several errors. So did Krzysztof Bolejko, who was the first reader of this text even before it was typed into a computer file. J. P. acknowledges the support from the Consejo Nacional de Ciencia y Teconología projects 32427E and 41993F.

In preparing the second edition, the second author (A. K.) received valuable help from Leszek Sokołowski concerning the variational principle for the Einstein–Maxwell equations. Mauro Carfora and Bob Jantzen helped with locating the original papers on the Fermi–Walker transport. Marie-Noelle Célérier helped with finding references to Michel Chasles papers. Neil Ashby kindly sent me the eps files of the figures illustrating his papers and allowed me to use them in this book. Consultations by Kayll Lake, George Ellis, Roy Maartens and Mikołaj Korzyński are also gratefully acknowledged.