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Definition and Properties of the GFF

1.1 Discrete case

Note: The discrete case is included here only for the purpose of guiding intuition when we come to work in the continuum.

Consider a finite, weighted, undirected graph $\mathcal{G} = (V, E)$ (with weights $(w_e)_{e \in E}$ on the edges). For instance, $\mathcal{G}$ could be a finite portion of the Euclidean lattice $\mathbb{Z}^d$ with weights $w_e \equiv 1$. Let ∂ be a distinguished set of vertices, called the boundary of the graph, and set $\hat{V} = V \setminus \partial$. Let $(X_t)_{t \geq 0}$ be the random walk on $\mathcal{G}$ in continuous time, meaning that it jumps from x to y at rate $w_{x,y}$, and let τ be the first time that X hits ∂ (which we assume to be finite almost surely for every starting point).

Write $Q = (q_{x,y})_{x,y \in V}$ for the Q -matrix of X . That is, its infinitesimal generator, so that for each $x \in V$, $q_{x,y} = w_{x,y}$ for $y \neq x$ and $q_{x,x} = -\sum_{y \sim x} w_{x,y} < \infty$, where $y \sim x$ means that x and y are connected by an edge in E . Note that the uniform measure $\pi(x) \equiv 1$ is reversible for X . We write $\mathbb{P}_x$ for the law of the random walk started at $x \in V$ and $\mathbb{E}_x$ for the corresponding expectation.

Definition 1.1 (Green function) The Green function $G(x, y)$ is defined for any $x, y \in V$ by setting

$$G(x, y) = \mathbb{E}_x \left(\int_0^\infty \mathbf{1}_{\{X_t = y; \tau > t\}} dt \right).$$

In other words, $G(x, y)$ is the expected time that X spends at y , when started from x , before hitting the boundary. Note that with this definition, we have $G(x, y) = G(y, x)$ for all $x, y \in \hat{V}$, since $\mathbb{P}_x(X_t = y; \tau > t) = \mathbb{P}_y(X_t = x; \tau > t)$ by reversibility of X with respect to π .

An equivalent expression for the Green function when working with the random walk in discrete time $Y = (Y_n)_{n \geq 0}$ (which jumps from x to y with probability proportional to $w_{x,y}$) is

$$G(x, y) = \frac{1}{q_y} \mathbb{E}_x \left(\sum_{n=0}^{\infty} \mathbf{1}_{\{Y_n=y; \tau(Y)>n\}} \right), \quad (1.1)$$

where $q_y = \sum_{y \sim x} w_{x,y} = -q_{y,y}$ and $\tau(Y)$ is the first time that Y hits ∂D . Indeed, X and Y can be coupled in such a way that for each $y \in \hat{V}$ and each visit of Y to y , X stays at y for an exponentially distributed time with mean $1/q_y$.

The Green function is a basic ingredient in the definition of the Gaussian free field, so the following elementary properties will be important to us.

Proposition 1.2 *Let $\hat{Q}$ denote the restriction of Q to $\hat{V} \times \hat{V}$. Then*

1. $(-\hat{Q})^{-1}(x, y) = G(x, y)$ for all $x, y \in \hat{V}$.
2. G is a symmetric and non-negative definite function. That is, one has

$$G(x, y) = G(y, x)$$

for all $x, y \in V$, and if $(\lambda_x)_{x \in V}$ is any vector of length $|V|$, then

$$\sum_{x, y \in V} \lambda_x \lambda_y G(x, y) \geq 0.$$

Equivalently, G is symmetric and therefore diagonalisable (when viewed as a matrix), and all of the eigenvalues of G are non-negative. Furthermore, restricted to $\hat{V}$, G is a positive definite function (i.e. its eigenvalues are strictly positive).

3. $G(x, \cdot)$ is discrete harmonic in $\hat{V} \setminus \{x\}$; more precisely G is the unique function of $x, y \in V$ such that $\hat{Q}G(x, \cdot) = -\delta_x(\cdot)$ for all $x \in \hat{V}$, and satisfies the “boundary condition” $G(x, \cdot) = 0$ on ∂ for all $x \in V$.

Here, $\delta_x(\cdot)$ denotes the Dirac function at x , namely $\delta_x(\cdot) = 1_{\{\cdot=x\}}$. We also use the natural notation $Qf(x) = \sum_{y \sim x} q_{xy}(f(y) - f(x))$ for the action of the generator Q on functions. Viewed as an operator in this way, Q is often referred to as the discrete Laplacian in continuous time. (Note that by definition, $Qf(x)$ measures the infinitesimal expected change in $f(X_t)$ if the chain starts at x .)

Remark 1.3 The proof below is written in the formalism of continuous time Markov chains, which is a little more natural. However, it can equivalently be written using discrete-time Markov chains and the definition of the Green function in (1.1).

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Proof Note that since $\hat{Q}$ is symmetric, it is diagonalisable, and that all its eigenvalues are negative (this is true of the infinitesimal generator of any Markov chain in continuous time, and here $\hat{Q}$ is nothing else but the infinitesimal generator of the Markov chain absorbed at ∂). Since the chain is absorbed at ∂ , 0 is not an eigenvalue and all the eigenvalues of $\hat{Q}$ are therefore strictly negative.

Furthermore, if $\hat{P}^t(x, y) = \mathbb{P}_x(X_t = y, \tau > t)$, then $\hat{P}^t$ satisfies the backward Kolmogorov equation, namely

$$(d/dt)\hat{P}^t(x, y) = \hat{Q}\hat{P}^t(x, y),$$

so that $\hat{P}^t(x, y) = e^{\hat{Q}t}(x, y) = 1 + \sum_{j \geq 1} \frac{1}{j!} (\hat{Q})^j(x, y)$. It then follows, by Fubini, that

$$\begin{aligned} G(x, y) &= \mathbb{E}_x \left(\int_0^\infty \mathbf{1}_{\{X_t=y; \tau>t\}} dt \right) \\ &= \int_0^\infty \hat{P}^t(x, y) dt \\ &= \int_0^\infty e^{\hat{Q}t}(x, y) dt \\ &= (-\hat{Q})^{-1}(x, y). \end{aligned} \tag{1.2}$$

The justification for the last equality comes from thinking about the action of the operator $\int_0^\infty e^{\hat{Q}t} dt$ on a single eigenfunction of $\hat{Q}$ (recalling that the corresponding eigenvalue is negative). Since there is a basis of eigenfunctions of $\hat{Q}$ by symmetry, this suffices to prove the last equality.

For the second point, we have already mentioned that $G(x, y) = G(y, x)$. Since $G(x, y) = 0$ whenever $y \in \partial$ it suffices to show that the restriction of G to $\hat{V}$ is positive definite. For this, we can use again that all the eigenvalues of $-\hat{Q}$, and hence of $(-\hat{Q})^{-1}$ are positive. This gives that G is positive definite when restricted to $\hat{V}$, by (1.2).

Let us finally check the third point. This can be seen as a straightforward consequence of the first point, but we prefer to also include a probabilistic proof which is based on the Markov property; effectively, we decompose according to the first jump of the chain.¹ Let $L(x) = \int_0^\infty \mathbf{1}_{\{X_t=x; \tau>t\}} dt$. Suppose that $y \neq x$ and $t \geq 0$. If $X_0 = y$ and J is the first time that X jumps away from y (so J is an exponential random variable with rate $q_y = -q_{y,y}$), we can decompose $\mathbb{E}_y(L(x))$ according to whether $\{J > t\}$ or $\{J = s \text{ for some } 0 \leq s \leq t\}$. Also

¹ The analogous derivation of the same fact using the discrete-time chain Y instead of the continuous time chain X is in fact slightly simpler – we recommend this as an exercise for the reader!

applying the Markov property at time J , we obtain that

$$\begin{aligned} G(x, y) &= G(y, x) = \mathbb{E}_y(L(x)) \\ &= \mathbb{E}_y(L(x) | J > t) \mathbb{P}_y(J > t) \\ &\quad + \int_0^t \sum_{z \neq y} \mathbb{P}_y(J \in ds, X_J = z) \mathbb{E}_y(L(x) | J = s, X_J = z) \\ &= G(y, x) e^{-q_y t} + \int_0^t q_y e^{-q_y s} ds \sum_{z \neq y} \frac{q_{y,z}}{q_y} \mathbb{E}_z(L(x)). \end{aligned}$$

Taking the time derivative on both sides at $t = 0$ and again invoking symmetry, we arrive at the equality

$$0 = -q_y G(y, x) + \sum_{z \neq y} q_{y,z} G(z, x) = -q_y G(x, y) + \sum_{z \neq y} q_{y,z} G(x, z).$$

This means (for fixed x , viewing $G(x, y)$ as a function $g(y)$ of y only) that $\mathcal{Q}g(y) = \sum_z q_{y,z} g(z) = 0$. Hence $G(x, \cdot)$ is harmonic in $\hat{V} \setminus \{x\}$.

When $y = x$, a similar argument can be made, but now the event $\{J > t\}$ contributes to $L(x)$, namely,

$$\begin{aligned} G(x, x) &= \mathbb{P}(J > t)(t + G(x, x)) + \int_0^t q_x e^{-q_x s} ds \sum_{z \neq x} \frac{q_{x,z}}{q_x} \mathbb{E}_z(L(x)) \\ &= e^{-q_x t}(t + G(x, x)) + \int_0^t e^{-q_x s} \sum_{z \neq x} q_{x,z} G(x, z) ds. \end{aligned}$$

Taking the derivative of both sides at $t = 0$ gives

$$0 = -q_x G(x, x) + 1 + \sum_{z \neq x} q_{x,z} G(x, z),$$

and hence

$$\sum_z q_{xz} G(x, z) = -1.$$

The uniqueness comes from the invertibility of $-\hat{\mathcal{Q}}$. □

Remark 1.4 An alternative proof of the first point (i.e. of (1.2)) uses the transition matrix $\hat{R}^n(x, y) = \mathbb{P}_x(Y_n = y, \tau(Y) > n)$ of the jump chain. Indeed, we have already noted that

$$\begin{aligned} G(x, y) &= \frac{1}{q_y} \mathbb{E}_x \left(\sum_{n=0}^{\infty} \mathbf{1}_{\{Y_n = y, \tau(Y) > n\}} \right) \\ &= \frac{1}{q_y} \sum_{n=0}^{\infty} \hat{R}^n(x, y) \end{aligned}$$

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$$\begin{aligned}
 &= \frac{1}{q_y} (I - \hat{R})^{-1}(x, y) \\
 &= (-\hat{Q})^{-1}(x, y),
 \end{aligned}$$

where in jumping from the second to the third line, we used the fact that $\sum_{n=0}^{\infty} \hat{R}^n = (I - \hat{R})^{-1}$, an identity valid for any matrix of spectral radius (i.e. largest eigenvalue modulus) strictly smaller than 1, which is the case here. An alternative proof that G is non-negative definite can be obtained using the same argument in the proof of Lemma 1.28 (this is stated in the continuous case but can also easily be adapted to this discrete setting).

Definition 1.5 (Discrete GFF) The (zero boundary) discrete Gaussian free field on $\mathcal{G} = (V, E)$ is the centred Gaussian vector $(h(x))_{x \in V}$ with covariance given by the Green function G .

Remark 1.6 This definition is justified. Indeed, suppose that $(C(x, y))_{x, y \in V}$ is a given function. Then there exists a centred Gaussian vector X having covariance matrix C if and only if C is symmetric and non-negative definite (in the sense of property 2 above).

Note that if $x \in \partial$, then $G(x, y) = 0$ for all $y \in V$ and hence $h(x) = 0$ almost surely.

In fact, it is possible to provide a concrete construction of the discrete Gaussian free field, in terms of i.i.d. standard Gaussian random variables. This construction has the advantage that it is very easy to implement on a computer to produce simulations, such as the one in Figure 1.1. We first introduce some notations. Set $N = |\hat{V}|$ and consider the space of functions $f: \hat{V} \rightarrow \mathbb{R}$, equipped with the inner product

$$(f, g) = \sum_{x \in \hat{V}} f(x)g(x). \quad (1.3)$$

For this reason (and even though $\hat{V}$ is finite), we denote this space of functions by $\ell^2(\hat{V})$. Any function in $\ell^2(\hat{V})$ can canonically be extended to a function on V by setting it to zero on ∂ . Recall that a function $f \in \ell^2(\hat{V})$ is an eigenfunction of $-\hat{Q}$ with eigenvalue λ (necessarily positive) if for all $x \in \hat{V}$, $-\hat{Q}f(x) = \lambda f(x)$, that is,

$$-\sum_{y \in \hat{V}} q_{x,y} f(y) = \lambda f(x).$$

As already mentioned in the proof of Proposition 1.2, since $-\hat{Q}$ is symmetric, it is diagonalisable in an orthonormal basis of $\ell^2(\hat{V})$. Let $e_1, \dots, e_N$ denote the

orthonormal eigenfunctions and let $0 < \lambda_1 \leq \dots \leq \lambda_N$ denote the corresponding eigenvalues (with multiplicities).

Theorem 1.7 *Let $(e_m)_{m=1}^N$ and $(\lambda_m)_{m=1}^N$ be as above. Then for $x, y \in \hat{V}$ we have the expansion*

$$G(x, y) = \sum_{m=1}^N \frac{1}{\lambda_m} e_m(x) e_m(y). \quad (1.4)$$

This also extends to ∂ if we extend e_m by zero on ∂ .

Furthermore, let $(X_m)_{m=1}^N$ be a sequence of i.i.d. standard Gaussians. Set

$$h(x) := \sum_{m=1}^N \frac{X_m}{\sqrt{\lambda_m}} e_m(x); \quad x \in V \quad (1.5)$$

Then h is a discrete GFF on $\mathcal{G}$.

Proof Since $-\hat{Q}$ is invertible, it suffices to check that for $x \in \hat{V}$, if we define $g(y) = g_x(y)$ as on the right-hand side of (1.4), namely $g(y) = \sum_{m=1}^N (1/\lambda_m) e_m(x) e_m(y)$ viewed as a function of $y \in \hat{V}$, one has

$$-\hat{Q}g = \delta_x. \quad (1.6)$$

By linearity and since e_m is an eigenfunction of $-\hat{Q}$, we see that (recall that $x \in \hat{V}$ is fixed)

$$-\hat{Q}g = \sum_{m=1}^N \frac{1}{\lambda_m} e_m(x) (-\hat{Q}e_m) = \sum_{m=1}^N e_m(x) e_m.$$

On the other hand, expanding δ_x in the basis $(e_m)_{m=1}^N$, we also find that

$$\delta_x = \sum_{m=1}^N (\delta_x, e_m) e_m = \sum_{m=1}^N e_m(x) e_m,$$

which is indeed the same as the right-hand side of the previous equation. This proves (1.6) and thus (1.4).

Turning to (1.5), we simply note that $(h(x))_{x \in \hat{V}}$ is clearly a centred Gaussian vector, whose covariance is given by

$$\begin{aligned} \mathbb{E}[h(x)h(y)] &= \mathbb{E} \left(\sum_{m, m'=1}^N \frac{X_m X_{m'}}{\sqrt{\lambda_m \lambda_{m'}}} e_m(x) e_{m'}(y) \right) \\ &= \sum_{m=1}^N \frac{1}{\lambda_m} e_m(x) e_m(y) = G(x, y) \end{aligned}$$

by (1.4), as desired. $\square$

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Figure 1.1 A discrete Gaussian free field. Simulation by Oskar-Laurin Koerner.

Usually for Gaussian fields, looking at the covariance structure is the most useful way of gaining intuition. However, in this case, the joint probability density function of the $|V|$ components of h is perhaps more illuminating.

Theorem 1.8 (Law of the GFF and Dirichlet energy) *The law of the discrete GFF is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^{\hat{V}}$, with joint density proportional to*

$$\exp\left(-\frac{1}{4} \sum_{x,y \in V} q_{x,y} (h(x) - h(y))^2\right)$$

at any point $(h(x))_{x \in V}$ with $h(x) = 0$ for $x \in \partial$, viewed as a fixed element of $\mathbb{R}^{\hat{V}}$. (Note that the sum includes the vertices $v \in \partial$.)

Remark 1.9 The previous formula might seem a little confusing at first, since we are using $(h(x))_{x \in \hat{V}}$ both to denote the random vector consisting of the values of the discrete Gaussian free field, and for a fixed (deterministic) element in $\mathbb{R}^{\hat{V}}$ at which we evaluate the density of this random vector. To avoid any confusion, the formula above means the following: if we write Y_v for the random variable $Y_v := h(v)$, then

$$\mathbb{P}((Y_v)_{v \in V} \in A) = \int_A \frac{1}{Z} \exp\left(-\frac{1}{4} \sum_{v,w \in V} q_{v,w} (x_v - x_w)^2\right) \prod_{v \in \hat{V}} dx_v,$$

where $Z = \int_{\mathbb{R}^N} \exp\left(-\frac{1}{4} \sum_{v,w \in V} q_{v,w} (x_v - x_w)^2\right) \prod_{v \in \hat{V}} dx_v$, where $N = |\hat{V}|$. This holds for any Borel set A contained in the hyperplane $\{(x_v)_{v \in V} : x_v = 0 \text{ for all } v \in \partial\}$ of $\mathbb{R}^V$.

For a given function $f : V \rightarrow \mathbb{R}$, the quantity

$$\mathcal{E}(f, f) := \frac{1}{2} \sum_{x,y \in V} q_{x,y} (f(x) - f(y))^2 \quad (1.7)$$

is known as the **Dirichlet energy** of f : it is a discrete analogue of $(1/2) \int_D |\nabla f|^2$.

Proof of Theorem 1.8 The result follows from the fact that for a centred Gaussian vector $(Y_1, \dots, Y_N)$ with invertible covariance matrix Σ , the joint probability density function on $\mathbb{R}^N$ is proportional to

$$f(x_1, \dots, x_N) = \exp\left(-\frac{1}{2}x^T \Sigma^{-1}x\right).$$

For us, the vertices $v \in \hat{V}$ play the roles of the indices $1 \leq i \leq N$ above with $N = |\hat{V}|$, and the values $h(v)$ for $v \in V$ play the roles of the x_i (to get a non-degenerate covariance matrix we restrict ourselves to vertices in $\hat{V}$, in which case G is invertible by Proposition 1.2). Note that since we are only considering h with $h(v) = 0$ for $v \in \partial$, it suffices to show that

$$-\frac{1}{2}h(\hat{v})^T G^{-1}h(\hat{v}) = -\frac{1}{4} \sum_{x,y \in V} q_{x,y}(h(x) - h(y))^2, \text{ for } h(\hat{v}) = (h(v))_{v \in \hat{V}}.$$

Recall that $(-\hat{Q})^{-1}(x, y) = G(x, y)$ for $x, y \in \hat{V}$, so that $G^{-1}(x, y) = -q_{xy}$. Hence

$$h(\hat{v})^T G^{-1}h(\hat{v}) = \sum_{x,y \in \hat{V}} G^{-1}(x, y)h(x)h(y) = \sum_{x,y \in \hat{V}} -q_{x,y}h(x)h(y).$$

Moreover, as we only consider h with $h(x) = 0$ for $x \in \partial$, this can be rewritten as

$$\begin{aligned} - \sum_{x,y \in V} q_{x,y}h(x)h(y) &= \frac{1}{2} \sum_{x,y \in V} q_{x,y}(h(x) - h(y))^2 - \frac{1}{2} \sum_{x,y \in V} h(x)^2 q_{x,y} \\ &\quad - \frac{1}{2} \sum_{x,y \in V} h(y)^2 q_{x,y}, \end{aligned}$$

where since $\sum_{y \in V} q_{x,y} = 0$ and $q_{x,y} = q_{y,x}$ for all x, y , the terms

$$\sum_{x,y \in V} h(x)^2 q_{x,y} \text{ and } \sum_{x,y \in V} h(y)^2 q_{x,y}$$

are both equal to 0. Note that in this final line of reasoning, it is important to sum over all of V and not just $\hat{V}$. Thus

$$-\frac{1}{2}h(\hat{v})^T G^{-1}h(\hat{v}) = -\frac{1}{2} \times \frac{1}{2} \sum_{x,y \in V} q_{x,y}(h(x) - h(y))^2,$$

as required. $\square$

Notice that the Dirichlet energy of functions is minimised by harmonic functions. This means that the Gaussian free field can be viewed as a “Gaussian

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perturbation of a harmonic function”: as much as possible, it “tries” to be harmonic. In fact, this is a little ironic, given that in the continuum it is not even a function (see Sections 1.3 and 1.7).

This heuristic is at the heart of the Markov property, which is without a doubt the most useful property of the GFF. We state it here without proof, as we will soon prove its (very similar) continuum counterpart.

Theorem 1.10 [Markov property of the discrete GFF] Fix $U \subset V$. The discrete GFF $h = (h(x))_{x \in V}$ can be decomposed as

$$h = h_0 + \varphi,$$

where h_0 is Gaussian free field on U and φ is harmonic in U . Moreover, h_0 and φ are independent.

By a Gaussian free field in U , we mean the GFF on the graph (V, E) but now with $\partial = V \setminus U$, in particular $h_0 = 0$ outside of U .

In other words, this theorem says that conditionally on the values of h outside of U , the field can be written as the sum of two independent terms. One of these is a zero boundary GFF in U , and the other is just the harmonic extension into U of the values of h outside U . To see this, note that the information about the values of h outside of U is completely contained in φ , since h_0 is zero outside of U . Thus conditioning on the values of h outside of U is the same as conditioning on φ . Since h_0 is independent of φ , the conditional law of h given φ is as described.

1.2 Continuous Green function

We will follow a route that is similar to the previous discrete case. First we need to recall the definition of the Green function. We will only cover the basics here, and readers who want to know more are advised to consult, for instance, Lawler’s book [Law05] which reviews important facts in a very accessible way. The presentation here will be somewhat different.

Let $d \geq 1$. Let $p_t(x, y)$ denote the transition probability of a Brownian motion B in $\mathbb{R}^d$ with “speed” two (i.e. $B_t = (X_{2t}^1, \dots, X_{2t}^d)$ for $t \geq 0$, where $X^1, \dots, X^d$ are independent standard Brownian motions² in $\mathbb{R}$). Then

$$p_t(x, y) = (4\pi t)^{-d/2} \exp(-|x - y|^2 / (4t)), \quad (1.8)$$

which by the Markov property is also the density, with respect to Lebesgue

² This choice ensures that the infinitesimal generator of B is the Laplace operator Δ instead of $\Delta/2$.

measure on $\mathbb{R}^d$, of the law of B_t (when started from x). For $D \subset \mathbb{R}^d$ an open set, we define $p_t^D(x, y)$ to be the transition probability of Brownian motion with speed two, killed when leaving D , which is defined as the density, with respect to Lebesgue measure on $\mathbb{R}^d$, of the law of B_t , but restricted to the event $\{\tau_D > t\}$, where

$$\tau_D = \inf\{t > 0: B_t \notin D\}.$$

In other words, for any Borel set A in $\mathbb{R}^d$, it satisfies

$$\mathbb{P}_x(B_t \in A, \tau_D > t) = \int_{\mathbb{R}^d} 1_A(y) p_t^D(x, y) dy. \quad (1.9)$$

The (almost everywhere, for a fixed $t \geq 0$) existence of a function satisfying (1.9) follows directly from the Radon–Nikodym derivative theorem, since it is clear that if A has zero Lebesgue measure, then $\mathbb{P}_x(B_t \in A, \tau_D > t) \leq \mathbb{P}_x(B_t \in A) = 0$.

By conditioning on the position at time t of B_t , it is not hard to check that $p_t^D(x, y)$ can be expressed rather simply in terms of the whole space transition probabilities in (1.8) and the so-called (speed two) Brownian bridge³ $(b_s)_{0 \leq s \leq t}$ of duration t from x to y , which describes the law of B , conditionally given $B_0 = x$ and $B_t = y$. Namely, if we denote by $\mathbb{P}_{x \rightarrow y; t}$ this law, then we see that

$$\begin{aligned} \mathbb{P}_x(B_t \in A; \tau_D > t) &= \int_{\mathbb{R}^d} \mathbb{P}_{x \rightarrow y; t}(b_t \in A; \tau_D > t) p_t(x, y) dy \\ &= \int_{\mathbb{R}^d} 1_A(y) \mathbb{P}_{x \rightarrow y; t}(\tau_D > t) p_t(x, y) dy. \end{aligned}$$

Comparing with (1.9), we deduce that, for every fixed $t \geq 0$ and almost every y ,

$$p_t^D(x, y) = \pi_t^D(x, y) p_t(x, y); \text{ where } \pi_t^D(x, y) = \mathbb{P}_{x \rightarrow y; t}(\tau_D > t). \quad (1.10)$$

The right-hand side is easily seen to be a jointly continuous function in $t > 0$ and $x, y \in \overline{D}$, as this is clearly satisfied by both $\pi_t^D(x, y)$ and $p_t(x, y)$ separately. This defines the transition probability function $p_t^D(x, y)$ of Brownian motion killed when leaving D uniquely.

Clearly, by the Markov property of Brownian motion, the transition probabilities satisfy the Chapman–Kolmogorov equation:

$$p_{t+s}^D(x, y) = \int_{\mathbb{R}^d} p_t^D(x, z) p_s^D(z, y) dz \quad \text{for } s, t \geq 0, x, y \in D. \quad (1.11)$$

Note also immediately for future reference that, by definition of $p_t^D(x, y)$

³ A reader who is unfamiliar with the notion of Brownian bridge may without danger skip to the conclusion immediately following (1.10).