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978-1-009-40238-5 — Electroweak Theory
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Excerpt
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1

The electromagnetic current and its properties

1.1 Introduction

The theory of the weak interactions, better known as the electroweak theory, was developed in two stages. In the first stage, a phenomenological interaction was introduced and was extended when additional experimental results became available. At that stage a large number of observations could be accounted for by empirical rules. There still remained the desire to develop a basic theory that was finite and renormalizable. This was achieved in the second stage by combining the electromagnetic and weak interactions into a gauge theory – the electroweak theory.

The effective current–current interaction was introduced by Fermi in 1934,

$$\mathcal{H}_{\text{eff}} = -\frac{G_{\text{F}}}{\sqrt{2}} J_\mu(x) J^{\mu\dagger}(x), \tag{1.1}$$

and was responsible for charged-current weak interactions of leptons and hadrons. The current was originally introduced, in analogy to electrodynamics, for the interaction of the electron with its neutrino and also for the neutron–proton transition

$$J_\mu(x) = \overline{\Psi}_{\nu_e}\gamma_\mu(1-\gamma_5)\Psi_{\text{e}} + \overline{\Psi}_{\text{p}}\gamma_\mu(1-\gamma_5)\Psi_{\text{n}} + \overline{\Psi}_{\nu_\mu}\gamma_\mu(1-\gamma_5)\Psi_\mu + \cdots. \tag{1.2}$$

Here the Ψs are the fields of the fermions and the γs are the Dirac γ-matrices in the notation of Bjorken and Drell (1965). The shortcoming of this theory is known as the unitarity problem and shows up in many reactions. For example, for the reaction

$$\nu_\mu + \text{e}^- \longrightarrow \nu_\text{e} + \mu^-$$

we can calculate the cross section, which to lowest order is

$$\sigma_{\text{tot}}(\nu_\mu \text{e}^- \to \nu_\text{e}\mu^-) = \frac{G_\text{F}^2 s}{\pi} \tag{1.3}$$

with $s = 4E_{\text{cm}}^2$, where terms proportional to the masses of the leptons have been omitted at high energies. Because of the point coupling in (1.1) only the lowest partial wave (angular momentum zero) can contribute to the scattering amplitude. Then conservation of probability (unitarity) in quantum mechanics requires (see Problems 1 and 2 at the end of Chapter 2)

$$\sigma_{\text{inelastic}}^{l=0} \leq \frac{\pi}{2E_{\text{cm}}^2} \tag{1.4}$$

for any scattering process. From (1.3) and (1.4) we find that the theory is consistent with unitarity only for

$$E_{\text{cm}} \leq \left(\frac{\pi\sqrt{2}}{4G_{\text{F}}}\right)^{\frac{1}{2}} = 309\,\text{GeV}. \tag{1.5}$$

Thus the theory is incomplete.

On the other hand, why should we believe the first-order-term result for such high energies? It is not a matter of belief but an unfortunate fact of life that we cannot calculate higher-order contributions. The theory, which is based on the Hamiltonian (1.1), is non-renormalizable and does not allow a well-defined perturbation expansion.

At this point we fall back upon the most successful field theory at our disposal: quantum electrodynamics (QED). We describe in this chapter its salient features and we try to develop in Part II of this book, in analogy to QED, a gauge theory of weak and electromagnetic interactions. In fact the second stage in the development of the weak interactions is to construct a well-defined and renormalizable theory.

We start with the Dirac Lagrangian for an electron interacting with the electromagnetic field,

$$\mathcal{L} = \overline{\Psi}\left(\mathrm{i}\gamma^\mu \frac{\partial}{\partial x^\mu} + e\gamma^\mu A_\mu - m\right)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{1.6}$$

We think of Ψ as the electron field whose current

$$j_\mu = \overline{\Psi}(x)\gamma_\mu\Psi(x) \tag{1.7}$$

interacts with the electromagnetic field

$$\mathcal{L}_{\text{F}} = \overline{\Psi}(\mathrm{i}\gamma^\mu\partial_\mu - m)\Psi + ej^\mu A_\mu. \tag{1.8}$$

The interaction term $e\overline{\Psi}\gamma_\mu\Psi A^\mu$ fixes the vertex and the electron propagator is the inverse of the kinetic term.

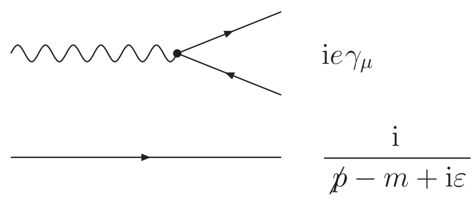

Figure 1.1. The photon–fermion vertex and the propagator.

Finally, the last term in (1.6) gives the interaction between photons and involves the electromagnetic field tensor

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}. \tag{1.9}$$

Gauge invariance forbids a term $m_\gamma^2 A_\mu A^\mu$ that would give a mass to the photon. QED has been one of the most precise and successful theories in all of physics and has been tested to a few parts per million.

As mentioned above, the electromagnetic current describes the interaction of the photon with a charged fermion. The current is a local operator

$$j_\mu(x) = \overline{\Psi}_l(x)\gamma_\mu\Psi_l(x), \tag{1.10}$$

where $\Psi_l(x)$ is the field for the lepton l and γ_μ is a Dirac matrix. The current $j_\mu(x)$ is a generalization of the classical concept of a current as it appears in Maxwell's theory. In classical electrodynamics $j_\mu(x)$ is a four-vector with components

$$\begin{aligned} j^\mu(x) &= \left[c\rho(x),\, \vec{j}(x) = \rho(x)\vec{v}\right] \\ &= \rho(x)\left[c, \vec{v}\,\right], \end{aligned} \tag{1.11}$$

with $\rho(x)$ denoting the charge density, the vector $\vec{j}(x)$ the charge flow, c the speed of light, and $\vec{v}$ the velocity of the charge density. The total charge of a particle is given by the integral

$$c\,Q = \int d^3x\; j_0(x). \tag{1.12}$$

The current in (1.11) is an operator that transforms like a four-vector. The fields occurring above are also operators that create and destroy localized particle states. They satisfy canonical commutation relations, which quantize the theory. The computational methods of QED can be found in many books given in the references. We shall assume that the reader is familiar with the methods of quantum electrodynamics.

1.2 The current for hadronic states

The electromagnetic current for a proton is more complicated since protons are not point-like particles, but have a measurable physical size formed by the cloud of pions and other hadrons which surrounds them. As a first attempt one would write the electromagnetic current for a proton in terms of free fields,

$$J_\mu = \overline{\Psi}_{\mathrm{p}'}(x)\gamma_\mu \Psi_{\mathrm{p}}(x) = u\overline{(p')}\gamma_\mu u(p)\mathrm{e}^{\mathrm{i}(p'-p)x}. \tag{1.13}$$

This form is ruled out immediately because it describes a point particle with unit charge and a Dirac magnetic moment. It obviously fails for the case of a proton, which has size and an anomalous magnetic moment. This implies a charge distribution and requires additional terms on the right-hand side.

One therefore expects a more general structure, which is introduced by considering the hadronic current as a vector operator that satisfies general symmetry principles. We begin by considering the interaction of the electromagnetic field $A_\mu(x)$ with protons. The matrix element contains the term

$$\langle p'|J_\mu(x)\mathrm{e}^{-\mathrm{i}qx}|p\rangle. \tag{1.14}$$

Under translations in space and time $J_\mu(x)$ transforms as

$$J_\mu(x) = \mathrm{e}^{\mathrm{i}\hat{p}x} J_\mu(0)\mathrm{e}^{-\mathrm{i}\hat{p}x}, \tag{1.15}$$

where $\hat{p}$ is the operator of the four-momentum; thus (1.14) reduces to

$$\int \mathrm{d}^4x \langle p'|J_\mu(0)|p\rangle \mathrm{e}^{-\mathrm{i}(q+p-p')x} = \bar{u}(p')O_\mu(p', p)u(p) \int \mathrm{d}^4x\, \mathrm{e}^{-\mathrm{i}(q+p-p')x}, \tag{1.16}$$

with O_μ containing terms with γ-matrices, the antisymmetric tensor $\varepsilon_{\mu\nu\alpha\beta}$, and momenta. The spinors $u(p)$ and $u(p')$ are solutions of the free Dirac equation. These are the requirements of Lorentz invariance.

Two other properties are

(i) gauge invariance, which translates into

$$q^\mu \langle p'|J_\mu(0)|p\rangle = q^\mu \bar{u}(p')0_\mu u(p) = 0; \tag{1.17}$$

(ii) Hermiticity of the current

$$\begin{aligned} \langle p'|J_\mu(0)|p\rangle^* &= \langle p|J_\mu(0)|p'\rangle, \\ \left[\bar{u}(p')0_\mu u(p)\right]^+ &= \bar{u}(p)0_\mu u(p'), \end{aligned} \tag{1.18}$$

from which it follows that

$$O_\mu^+ = \gamma_0 O_\mu \gamma_0. \tag{1.19}$$

The above requirements limit the types of Dirac matrices and momenta which are included in the operator O_μ. The first subset of operators is

$$\left\{\ell_\mu = p_\mu + p'_\mu,\ q_\mu = p'_\mu - p_\mu,\ \gamma_\mu,\ \mathrm{i}\sigma_{\mu\nu}q^\nu,\ \sigma_{\mu\nu}\ell^\nu\right\}, \tag{1.20}$$

which appear in matrix elements of the vector current. In addition to the above operators, there are also others that contain γ_5 or the antisymmetric tensor. They are produced by higher-order weak or new interactions and their contributions to electromagnetic matrix elements are small. For completeness we include them here and discuss some properties in the next section. The second subset of matrices contains γ_5,

$$\left\{\ell_\mu\gamma_5,\ q_\mu\gamma_5,\ \gamma_\mu\gamma_5,\ \sigma_{\mu\nu}q^\nu\gamma_5\right\}, \tag{1.21}$$

and the third the antisymmetric tensor,

$$\left\{\varepsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}\ell_\nu,\ \varepsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}q_\nu,\ \varepsilon^{\mu\nu\alpha\beta}\gamma_\mu\gamma_5 q_\alpha\ell_\beta,\ \varepsilon^{\mu\nu\alpha\beta}\gamma_\nu q_\alpha\ell_\beta\right\}. \tag{1.22}$$

These terms are odd under parity transformations. Matrix elements of these operators are not all linearly independent. For instance, matrix elements of the last three terms in (1.22) are reduced, by judicious use of γ-matrix identities and the Dirac equation, to matrix elements of the second set (Nowakowski *et al.*, 2005).

The Gordon decomposition formula

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left(\frac{p'^\mu + p^\mu}{2m} + \frac{\mathrm{i}\sigma^{\mu\nu}q^\nu}{2m}\right)u(p) \tag{1.23}$$

eliminates one term in the first subset. Similarly, the term $\sigma_{\mu\nu}\ell^\nu$ can be replaced by $\bar{u}(p')q_\mu u(p)$. Thus the matrix element of the vector current has the general form

$$\bar{u}(p')\left(\gamma_\mu F_1(q^2) + \frac{\mathrm{i}\sigma_{\mu\nu}q^\nu}{2m}F_2(q^2) + q_\mu F_3(q^2)\right)u(p). \tag{1.24}$$

Gauge invariance gives an additional condition,

$$F_3(q^2) = 0. \tag{1.25}$$

The functions F_i with $i = 1, 2, 3$ are Lorentz scalars and their argument must remain unchanged under the replacement $p_\mu \to p_\mu + k_\mu$ and $p'_\mu \to p'_\mu + k_\mu$ with k_μ an arbitrary four-vector; consequently they are functions of $q^2 = (p' - p)^2$, which justifies the argument introduced in Eq. (1.24). We can use the Hermiticity condition as written in (1.19) to assure that the form factors are real functions. In summary, symmetry principles restrict the number and properties of the form factors. Some other consequences of symmetries are discussed in Chapter 2 and the problems given there.

What is the physical meaning of form factors? As the name indicates, they describe the structure or configuration of particles. Let us begin with an electron in the Dirac theory. To lowest order of electrodynamics $F_1(0) = 1$ and $F_2(0) = 0$. On replacing next the γ_μ term with the help of the Gordon decomposition, the coupling of the electron to the electromagnetic field $A_\mu(x)$ is written as

$$e\bar{\Psi}_{\rm f}(x)\gamma_\mu A^\mu(x)\Psi_i(x) = e\bar{\Psi}_{\rm f}(x)\left(\frac{p_\mu + p'_\mu}{2m} + \frac{{\rm i}\sigma_{\mu\nu}q^\nu}{2m}\right)\Psi_i(x)A^\mu(x). \quad (1.26)$$

The non-relativistic limit produces two terms. The first term, from the sum of momenta,

$$e\bar{u}(p')u(p)A^0(x) \quad (1.27)$$

couples the charge density to the scalar potential because the ratio of the three-momentum to the mass becomes very small. The second term couples the magnetic moment to an external magnetic field. Considering a constant magnetic field $\vec{B}$ and its potential $A_\mu(x)$, the interaction in configuration space is

$$\frac{e}{2m}\bar{\Psi}_{\rm f}(x)\sigma_{\mu\nu}\Psi_i(x)\frac{\partial A_\mu(x)}{\partial x^\nu} = \bar{\Psi}_{A,{\rm f}}(x)\frac{e}{2m}\vec{\sigma}\cdot\vec{B}\,\Psi_{A,i}, \quad (1.28)$$

where Ψ_A are the upper components of the spinors (see Problem 2.5). The magnetic field is introduced as the rotation of the vector potential. Defining the magnetic moment as

$$\vec{\mu} = -g\frac{e}{2m}\vec{S} \quad \text{with} \quad \vec{S} = \frac{\vec{\sigma}}{2}, \quad (1.29)$$

we obtain for the electron the gyromagnetic ratio $g = 2$. Thus a Dirac electron has an intrinsic magnetic moment with the natural value of 2, which can be modified by radiative corrections.

Although we have started to derive a current for extended fermions, the results of this derivation in the form of Eqs. (1.27) and (1.29) are also valid for "point-like" particles, when higher-order electromagnetic corrections are taken into account. Indeed, the Lagrangian given in (1.6) will induce correction terms compatible with the symmetries of the Lagrangian. We see from (1.23) and (1.24) that F_2 will also contribute to the magnetic moment via $\mu = \frac{1}{2}[F_1(0) + F_2(0)]$. Both for the electron and for the muon, the magnetic moments have been measured very accurately. They have also been calculated theoretically and the agreement is very good. For the electron

$$\frac{1}{2}(g-2)_{\rm e} = 0.001\,159\,652\,209\,(31), \quad (1.30)$$

with the number in parentheses denoting the experimental accuracy. Very accurate results exist also for the muons. The deviation from the value of 2 comes from radiative corrections, which in quantum electrodynamics have been calculated precisely (Kinoshita, 1990).

The situation is very different for protons and neutrons. The experimental values are

$$F_1(0) = 1 \quad \text{and} \quad F_2(0) = 1.79 \qquad \text{for the proton,} \tag{1.31}$$

$$F_1(0) = 0 \quad \text{and} \quad F_2(0) = -1.91 \qquad \text{for the neutron.} \tag{1.32}$$

The changes come from the strong interactions and cannot yet be calculated. They are called the anomalous magnetic moments and have been measured in electron–hadron-scattering experiments. In addition to their values at $q^2 = 0$, the form factors have been measured over extended regions of the momentum-transfer squared and were found to decrease rapidly with q^2. This behavior indicates the existence of a charge distribution of virtual particles around the proton and the neutron, with the charge density decreasing rapidly with increasing radius. The motion of the particles creates magnetic fields, which are manifested in the values of the magnetic moments.

1.3 Parity-violating form factors

For completeness we include additional couplings of the photon induced by weak interactions inside the vertex. Omitting this section will not affect the study of the following chapters.

The electromagnetic force is not the only force between particles. For instance, the presence of weak terms changes the general structure of the electromagnetic matrix elements. The interaction of a photon with a particle does not mean that the whole process is electromagnetic, since higher-order corrections must also include the weak interactions. Conceptually it is easy to include these effects in the electromagnetic current, by dropping the restrictions that the current is invariant under the discrete symmetries charge conjugation C, parity P, and time-reversal T. Imposing Lorentz invariance, gauge invariance, and Hermiticity means that one must include two additional form factors (F_3 and F_4) and the electromagnetic current takes a more general form,

$$\bar{u}(p')0_\mu u(p) = \bar{u}(p')\Bigg[\gamma_\mu F_1(q^2) + \mathrm{i}\frac{\sigma_{\mu\nu}q^\nu}{2m}F_2(q^2) + \mathrm{i}\frac{\epsilon_{\mu\nu\alpha\beta}\sigma^{\alpha\beta}q^\nu}{4m}F_3(q^2) + \left(q_\mu - \frac{q^2}{2m}\gamma_\mu\right)\gamma_5 F_4(q^2)\Bigg]u(p). \tag{1.33}$$

We know from classical electrodynamics and quantum mechanics that the fields transform under parity P and time-reversal T as shown in Table 1.1.

Table 1.1

$\vec{B}$	$\xrightarrow{\text{P}}$	$\vec{B}$
$\vec{B}$	$\xrightarrow{\text{T}}$	$-\vec{B}$
$\vec{E}$	$\xrightarrow{\text{P}}$	$-\vec{E}$
$\vec{E}$	$\xrightarrow{\text{T}}$	$\vec{E}$
$\vec{\sigma}$	$\xrightarrow{\text{P}}$	$\vec{\sigma}$
$\vec{\sigma}$	$\xrightarrow{\text{T}}$	$-\vec{\sigma}$

From Table 1.1 we can infer immediately that $\vec{\sigma} \cdot \vec{B}$, an interaction defining the second form factor $F_2(q^2)$, conserves parity and time-reversal. Similarly, the non-relativistic reduction of all form factors including $F_3(q^2)$ and $F_4(q^2)$ is given by

$$\mathcal{H}_{\text{int}} \propto eA_0 - \mu\vec{\sigma} \cdot \vec{B} - d\vec{\sigma} \cdot \vec{E} - a\left[\vec{\sigma} \cdot \left(\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t}\right)\right], \tag{1.34}$$

with $F_1(0) = e$ (charge), $[F_1(0) + F_2(0)]/(2m) = \mu$ (magnetic dipole moment), $F_3(0)/(2m) = d$ (electric dipole moment), and $F_4(0) \propto a$ is called the anapole moment (Zeldovich, 1958). It is evident that the presence of F_3 leads to a parity- and time-reversal-violating interaction. Physical phenomena that exhibit violation of time-reversal are very scarce. Therefore, the observation of $d \neq 0$ will be a physical breakthrough. Up to now only upper limits for d have been established for electrons and nucleons.

The fourth form factor $F_4(q^2)$ is even under time-reversal but violates parity. It is frequently omitted from discussions of the electromagnetic form factors, because it is an off-shell form factor, in the sense that its interaction with an on-shell photon vanishes. This is easily seen because $q^2 = 0$ and $\varepsilon_\mu q^\mu = 0$ for on-shell photons. In addition, this form factor can appear only in matter with currents producing the electromagnetic fields, because for classical fields the expression $\vec{\nabla} \times \vec{B} - \partial \vec{E}/\partial t$, which appears in the anapole interaction, vanishes (Maxwell equation in vacuum) in the absence of a current. Finally, for neutral fermions, which do not carry any global quantum numbers, like Majorana neutrinos, only the anapole form factor is possible. For a more detailed treatment of the form factors I recommend Nowakowski *et al.* (2005).

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The weak currents

2.1 The weak currents and some of their properties

The effective weak interaction in Eq. (1.1) was motivated by nuclear β-decays. For many years this was the main theoretical framework for analyzing experiments. As new experimental discoveries became available, the form of the interaction was maintained but the current $J_\mu(x)$ was enlarged to incorporate the new observations. At the end of the sixties the charged current $J_\mu^\dagger(x)$ included a leptonic and a hadronic term,

$$J_\mu^\dagger = l_\mu^\dagger(x) + h_\mu^\dagger(x). \tag{2.1}$$

The leptonic part of the current is

$$l_\mu^\dagger(x) = \overline{\Psi}_{\mathrm{e}}(x)\gamma_\mu(1-\gamma_5)\Psi_{\nu_{\mathrm{e}}}(x) + \overline{\Psi}_\mu(x)\gamma_\mu(1-\gamma_5)\Psi_{\nu_\mu}(x), \tag{2.2}$$

with the first term corresponding to the electron and its neutrino and the second term to the muon and its neutrino. Its space-time structure has a vector part analogous to the electromagnetic current and an axial part introduced after the discovery of parity violation. A direct calculation using the currents in (2.2) gives the μ-decay spectrum, which is in good agreement with experiment. It also gives the decay rate of the muon as

$$\Gamma(\mu \to \mathrm{e} + \nu_{\mathrm{e}} + \bar{\nu}_\mu) = \frac{G_\mu m_\mu^5}{192\pi^3}. \tag{2.3}$$

From the observed decay rate and the mass of the muon the constant G_μ is determined to be

$$G_\mu = (1.166\,32 \pm 0.000\,04) \times 10^{-5}\,\mathrm{GeV}^{-2}. \tag{2.4}$$

This determination includes the effects of radiative corrections, which in the electroweak theory are finite and can be calculated precisely.