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Spin and helicity

Traditionally, in textbooks on quantum mechanics, spin is introduced via an idealized Stern–Gerlach experiment in which a non-relativistic beam of silver atoms passes through an inhomogeneous magnetic field. Each atom is treated as a single valence electron of charge $-e$ in an s-state. The subsequent splitting of the beam into two indicates the two-valuedness of s_z, which is related to the value $1/2$ for s, and the magnitude of the splitting shows that the magnetic moment $\boldsymbol{\mu}$ is related to $\mathbf{s}$ by

$$\boldsymbol{\mu} = -\frac{e}{mc}\mathbf{s},$$

the proportionality factor (the gyromagnetic ratio) being twice as big as the factor that classically gives the magnetic moment due to the orbital angular momentum of a point charge.

Historically, however, it seems that the early Stern–Gerlach experiments, begun in 1922, had no influence at all upon the discovery of spin, simply because they were too imprecise. Rather, the concept of spin appeared after a long and tedious battle to understand the splitting patterns and separations in line spectra. Several people had for various reasons discussed classical models of rotating charge distributions but Kronig, in 1924, was the first to show that an electron with spin 1/2 would explain the pattern of what we would today call $\mathbf{L}\cdot\mathbf{S}$ splitting, as well as anomalies in the Zeeman effect. He realized, though, that the gyromagnetic ratio $(-e/mc)$ needed for the latter would give $\mathbf{L}\cdot\mathbf{S}$ splittings twice as big as those observed. It is said that Pauli expressed his negative reaction to Kronig's idea with such vehemence that Kronig never published his work (Mehra and Rechenberg, 1982). Soon thereafter, in 1925, the same idea occurred to Uhlenbeck and Goudsmit (1925), who proceeded to a detailed analysis of the splittings, concluding at first that everything worked beautifully, but then becoming aware, as a consequence of a comment by Heisenberg, of the factor-of-2 inconsistency mentioned above.

Some months later Thomas demonstrated that a careful relativistic treatment produced exactly the factor of one half needed to bring about agreement between the theory of $\mathbf{L}\cdot\mathbf{S}$ splitting and experiment (Thomas, 1926).

In this work appears for the first time the infamous 'Thomas precession', which is mentioned, yet almost never explained, in all textbooks on quantum mechanics. We shall return to it later, but we should like, immediately, to demistify one aspect of it. It is usually said that relativistic effects produce a factor of one half. Now that would indeed be mysterious! What is forgotten is the fact that the $\mathbf{L}\cdot\mathbf{S}$ coupling is itself a relativistic effect. By means of a Lorentz transformation, we can understand that the electron, moving through the Coulomb field of the nucleus, sees a magnetic field in its rest frame. So the Thomas result is simply a correction to an already intrinsically relativistic effect.

1.1 Spin and rotations in non-relativistic quantum mechanics

In non-relativistic quantum mechanics the spin of a particle is introduced as an additional rotational degree of freedom. Analogously to orbital angular momentum one introduces three *spin operators*

$$\hat{\mathbf{s}} \equiv (\hat{s}_x, \hat{s}_y, \hat{s}_z);$$

the *spin states* $|sm\rangle$ are the simultaneous eigenstates of the commuting operators $\hat{\mathbf{s}}^2$ and $\hat{s}_z$, with eigenvalues $s(s+1)$ and m respectively. The *spin* s of the particle can be zero or a positive integer or half integer, while m can take values $-s \leq m \leq s$ in unit steps. The quantity m is referred to as the 'z-component of the spin'.

The three spin operators $\hat{s}_j$ satisfy the usual angular momentum commutation relations

$$\left[\hat{s}_j, \hat{s}_k\right] = i\epsilon_{jkl}\hat{s}_l. \tag{1.1.1}$$

For a free particle the spin degree of freedom is totally decoupled from the usual kinematic degrees of freedom, and this fact is implemented by writing the state vector in the form of a product, one factor referring to the usual degrees of freedom and the other to the spin degree of freedom. Thus, for a particle of momentum $\mathbf{p}$,

$$|\mathbf{p}; sm\rangle = |\mathbf{p}\rangle \otimes |sm\rangle \tag{1.1.2}$$

or, equivalently, for the wave function,

$$\psi_{\mathbf{p};sm}(\mathbf{x}) = \varphi_{\mathbf{p}}(\mathbf{x})\eta_{(m)} \tag{1.1.3}$$

where $\eta_{(m)}$ is a $(2s+1)$-component *spinor* and $\varphi_{\mathbf{p}}(\mathbf{x})$ is a standard Schrödinger wave function.

Since the labelling of the above spin states uses $m = \hat{s}_z$ and therefore makes reference to 'the z-direction' it is tacitly assumed that we are working in a well-defined, fixed coordinate reference system with origin O.

We wish now to discuss the effect of rotations upon the spin states. To begin with we recall the well-known rules for ordinary vectors. We shall denote by r the *physical operation* of a rotation. Thus, if we say that an object is rotated by e.g. $r_z(\theta)$, where θ is positive, then we mean that we are to physically push that object around the Z-axis through an angle θ in the sense of a right-hand screw advancing along OZ.

If we apply r to a given three-dimensional vector $\mathbf{A}$ we shall call the resultant rotated vector $r\mathbf{A}$ or $\mathbf{A}^r$. The action we have described is often referred to in the literature as the 'active' point of view as distinct from the 'passive' one, in which the axis system is rotated. We think that this is a confusing nomenclature. *All* our rotations *act* as described in the previous paragraph and if we wish to rotate axes we shall simply state that r acts on the coordinate axes.

The components of the rotated vector are related to the components A_i of $\mathbf{A}$ by

$$(r\mathbf{A})_i \equiv \mathbf{A}_i^r = R_{ij} A_j \tag{1.1.4}$$

where the 3×3 matrix R with elements R_{ij} depends, of course, on r. Strictly speaking, we should write it as $R(r)$. Sometimes it is convenient to write the components A_i in the form of a column vector

$$A = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}, \tag{1.1.5}$$

in which case (1.1.4) can be written in matrix notation as

$$A^r = RA. \tag{1.1.6}$$

As an example, if $r = r_y(\theta)$ then

$$R\left[r_y(\theta)\right] = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}. \tag{1.1.7}$$

For a tensor T, say of rank 2, the components of the rotated tensor T^r will be given by

$$T_{ij}^r = R_{ik} R_{jm} T_{km} \tag{1.1.8}$$

with obvious generalization to tensors of higher rank. It should be noted that tensors of rank ≥ 2 do not transform *irreducibly* under rotations. (The irreducible representations of the rotation group are discussed briefly in Appendix 1.)

Often one wishes to utilize a set of three orthogonal unit '*basis vectors*' $\mathbf{e}_{(i)}$ along the three coordinate axes. If we rotate one of them, say $\mathbf{e}_{(j)}$, the n components of $\mathbf{e}^r_{(j)}$ will be related to those of $\mathbf{e}_{(j)}$ by (1.1.4). But we can also consider $\mathbf{e}^r_{(j)}$ as a linear superposition of the $\mathbf{e}_{(i)}$, and one easily shows that

$$\mathbf{e}^r_{(j)} = R_{ij}\mathbf{e}_{(i)} = \left(\mathbf{R}^T\right)_{ji}\mathbf{e}_{(i)} \tag{1.1.9}$$

where R^T is the transpose of the matrix R. (Recall that for rotations R is orthogonal i.e. $R^T R = RR^T = I$.)

Note that whereas R appears in (1.1.4) it is R^T that occurs in (1.1.9).

We come now to the physical rôle of rotations. We are interested in the relationship between the descriptions that different observers give to the same physical phenomenon. Let $\mathbf{A}$ be a fixed vector, which observer O in our fundamental reference system S describes as having components A_j. Thus

$$\mathbf{A} = \sum_j A_j \mathbf{e}_{(j)} \tag{1.1.10}$$

Let O^r be an observer using a reference system S^r that has been rotated from S by a rotation r. Using the basis vectors $\mathbf{e}^r_{(l)}$ the observer describes $\mathbf{A}$ as having components $(A_l)_{S^r}$. Thus

$$\mathbf{A} = \sum_l (A_l)_{S^r}\, \mathbf{e}_{(l)} \tag{1.1.11}$$

and via (1.1.9) one finds, using $[R(r)]^{-1} = R(r^{-1})$, that

$$(A_i)_{S^r} = R_{ij}(r^{-1})A_j. \tag{1.1.12}$$

Although slightly misleading it is convenient to abbreviate (1.1.12) in the form

$$(A)_{S^r} = r^{-1}A. \tag{1.1.13}$$

In summary, if the *reference system* is rotated by r then the components of a *fixed vector*, as described in S^r and in S, are related via $R(r^{-1})$, in contradistinction to (1.1.4) in which R is shorthand for $R(r)$.

Spin-s spinors are dealt with in complete analogy to the above. We introduce $2s+1$ unit basis spinors $\eta_{(m)}$, where

$$\eta_{(s)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \eta_{(s-1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \ldots, \quad \eta_{(-s)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix};$$

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the $\eta_{(s)}$ represent eigenstates of $\hat{s}_z$. We write for a general spinor

$$\chi = \sum_m \chi_m \eta_{(m)}; \tag{1.1.14}$$

The numbers χ_m are the 'components' of χ. The components $(\chi_m)_{S^r}$ attributed to the spinor χ in the rotated reference frame S^r are related to χ_m analogously to (1.1.12):

$$(\chi_i)_{S^r} = \mathscr{D}^{(s)}_{ij}(r^{-1})\chi_j \tag{1.1.15}$$

where the matrices $\mathscr{D}^{(s)}(r)$ are the $(2s+1)$-dimensional representation matrices of the rotations r. (See Appendix 1; recall that the $\mathscr{D}$ are unitary matrices, i.e. $\mathscr{D}^\dagger\mathscr{D} = 1$.) By analogy with the inverse of (1.1.9) we have

$$\eta_{(m)} = \mathscr{D}^{(s)}_{m'm}(r^{-1})\eta^r_{(m')}. \tag{1.1.16}$$

The physical interpretation of (1.1.16) is that the state described by observer O in the frame S as $\eta_{(m)}$ is described by the rotated observer O' as a superposition of the states $\eta^r_{(m')}$.

Because of its importance we restate this in more general terms. If an observer O with reference system S sees a spin s particle in a state $|sm\rangle$ then the observer O^r whose reference frame S^r is rotated from S by the rotation r describes the state of the particle as $|sm\rangle_{S^r}$, where

$$|sm\rangle_{S^r} = \mathscr{D}^{(s)}_{m'm}(r^{-1})|sm'\rangle. \tag{1.1.17}$$

It is implicit in (1.1.17) that the states on the right-hand side are the $|sm\rangle$ of O^r.

Although it is not simple to see what we mean by physically rotating a spinor, by analogy with the vector case we shall talk about the active rotation of a state $|sm\rangle$ to $|sm\rangle^r$. Comparing with eqn (1.1.9) for the vector case, we shall interpret $|sm\rangle^r$ as given by

$$|sm\rangle^r = \mathscr{D}^{(s)}_{m'm}(r)|sm'\rangle. \tag{1.1.18}$$

It is very convenient in quantum mechanics to represent the effect of an operation by an *operator* acting directly on the state vectors. Thus we rewrite (1.1.18) in the form

$$|sm\rangle^r = U(r)|sm\rangle \tag{1.1.19}$$

where $U(r)$ is the *operator* representing the rotation r.

From (1.1.18) and (1.1.19) follows the well-known relation

$$\mathscr{D}^{(s)}_{m'm}(r) = \langle sm'|U(r)|sm\rangle. \tag{1.1.20}$$

In this operator notation (1.1.17) becomes

$$|sm\rangle_{S^r} = U(r^{-1})|sm\rangle. \tag{1.1.21}$$

In the case of spin 1/2, the spin operators $\hat{s}_j$ when acting on the two-dimensional spinors $\chi^{1/2}$ are represented by the set $\boldsymbol{\sigma}/2$ of 2×2 hermitian matrices $\sigma_j/2$, the σ_j being the usual Pauli matrices. In the case of arbitrary spin **s** the operators $\hat{s}_j$ when operating on the $(2s+1)$-dimensional spinor χ^s can similarly be represented by a set of three $(2s+1)$-dimensional hermitian matrices S_j, the S_j being the generalization of the Pauli matrices σ_j. There is an important and vital distinction, however, between the σ_j and the S_j, which in a sense makes the spin-1/2 case unique. It is a fact that the most general 2×2 hermitian matrix M can be specified by four independent real parameters and, as a consequence, because the σ_j are hermitian and independent, such a matrix M can always be written as

$$M = \tfrac{1}{2}\,(aI + \mathbf{b} \cdot \boldsymbol{\sigma}) \tag{1.1.22}$$

where the factor 1/2 is for convenience, I is the unit matrix, $\mathbf{b} \cdot \boldsymbol{\sigma}$ is short for $b_j\sigma_j$ and the four numbers a, b_j are all real. The form of (1.1.22) is particularly convenient since it is trivial to solve for a and b_j. One has

$$a = \mathrm{Tr}\ M, \qquad b_j = \mathrm{Tr}\ (\sigma_j M) \tag{1.1.23}$$

where Tr ≡ trace means the sum of the diagonal elements of the matrix.

The Pauli σ_j thus play a dual rôle. On the one hand, they represent the spin operators $\hat{s}_j$; on the other they furnish a basis for expressing any 2×2 hermitian matrix. *It is the confusion of these two rôles that sometimes leads to difficulties in understanding spin effects in relativistic situations.*

In the case of higher spin s the most general hermitian matrix is specified by $(2s+1)^2$ real parameters, so the set of the three S_j matrices is far from adequate as a basis for an expansion analogous to (1.1.22).

The special rôle of spin 1/2 shows itself in yet another way. The most general two-component spinor χ can be specified by four-real parameters, of which one, the overall phase, is totally irrelevant.

If, further, the spinor is normalized to unity, i.e.

$$\chi^\dagger \chi = 1,$$

we are left with two independent real parameters. Thus we can write, without loss of generality,

$$\chi = \begin{pmatrix} \cos\frac{1}{2}\theta\ e^{-i\phi/2} \\ \sin\frac{1}{2}\theta\ e^{i\phi/2} \end{pmatrix}. \tag{1.1.24}$$

If now we compute the *spin-polarization vector* $\boldsymbol{\mathcal{P}}_\chi$ defined by

$$\boldsymbol{\mathcal{P}}_\chi \equiv \langle \boldsymbol{\sigma} \rangle_\chi \equiv \chi^\dagger \boldsymbol{\sigma} \chi \tag{1.1.25}$$

we shall find that

$$\boldsymbol{\mathcal{P}}_\chi = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{1.1.26}$$

with $\mathcal{P}_\chi^2 = 1$. We see trivially that a knowledge of $\mathcal{P}_\chi$ completely specifies the quantum state χ. In the case of higher spin, one can still define a spin-polarization vector for a state χ such that

$$\mathcal{P}_\chi \equiv \langle \hat{\mathbf{s}} \rangle_\chi / s \equiv \mathbf{s}_\chi / s \tag{1.1.27}$$

where $\mathbf{s}$ is the *mean spin vector*, but now the three components of $\mathcal{P}$ are insufficient to fix the $2(2s+1) - 2$ independent parameters of the $(2s+1)$-dimensional spinor χ. Besides the case of spin 1/2 there is no other situation in nature where a knowledge of the spin-polarization vector completely specifies the quantum state. (Of course $\mathcal{P}$ and $\mathbf{s}$ are really pseudovectors. $\mathcal{P}$ is commonly referred to as the *polarization vector* but it is not at all the same thing as the polarization vector $\boldsymbol{\varepsilon}$ used in the description of photons or massive spin-1 particles. For this reason we shall refer to it as the *spin-polarization vector*.)

Finally we note a very important property of the matrices S_i representing the spin operator $\hat{s}_i$ for spin s, namely that they 'transform as vectors under rotation'. More precisely:

$$\mathscr{D}^{(s)}(r) S_i \mathscr{D}^{(s)\dagger}(r) = R_{ij}(r^{-1}) S_j. \tag{1.1.28}$$

This relation is best known in the spin-1/2 case in the simpler looking, but really equivalent, form

$$\langle \sigma_i \rangle_{S^r} = R_{ij}(r^{-1}) \langle \sigma_j \rangle \tag{1.1.29}$$

relating expectation values in S^r to those in S.

1.2 Spin and helicity in a relativistic process

The pioneering work of Dirac (1927) showed that spin emerges automatically in a relativistic theory and that it could no longer be treated as an independent additional degree of freedom. Nevertheless it is not trivial to see precisely how the spin is to be described relativistically, nor how it is to be interpreted physically. We shall give a brief discussion of this question, and then turn to consider the helicity states of Jacob and Wick (1959). Here our emphasis will be upon the physical interpretation and is somewhat complementary to the approach used by other authors.

We assume that the reader has some familiarity with homogeneous and inhomogeneous Lorentz transformations. A clear account can be found in Gasiorowicz (1967).

In a relativistic quantum theory the fundamental operators are the generators of the inhomogeneous Lorentz transformations. There are 10 of these. The three momentum operators $\hat{P}^j$ and the hamiltonian operator $\hat{P}^0$ generate translations in space and time respectively, and the six operators $\hat{M}^{\mu\nu} (= -\hat{M}^{\nu\mu})$ generate the homogeneous Lorentz transformations. It is physically more revealing to work not with the $\hat{M}^{\mu\nu}$ but with the combinations

$$\hat{J}_i \equiv -\tfrac{1}{2}\epsilon_{ijk}\hat{M}^{jk}, \qquad \hat{K}_i \equiv \hat{M}^{i0}, \tag{1.2.1}$$

which can be shown to be the generators of pure rotations and of pure Lorentz transformations ('boosts') respectively. Thus the $\hat{J}_i$ are identified as the total angular momentum operators.

As a consequence of the inherent characteristics of the inhomogeneous Lorentz transformations, one can derive commutation relations that must be satisfied by the generators. In particular, and in accordance with the interpretation of the $\hat{J}_i$ as angular momentum operators, one naturally finds

$$\left[\hat{J}_j, \hat{J}_k\right] = i\epsilon_{jkl}\hat{J}_l. \tag{1.2.2}$$

The operator $\hat{P}_\mu \hat{P}^\mu$ is invariant, i.e. it commutes with all the generators and its eigenvalues can thus be used to label states. Indeed, what we mean when we talk of an elementary particle of mass m is nothing other than matter that is an eigenstate of $\hat{P}_\mu \hat{P}^\mu$ with eigenvalue m^2.

The question that now arises is the following. If the theory already *contains* the spin then which operators are to be identified as the spin operators? Is there a set of operators $\hat{s}_i$, with commutation relations akin to eqn (1.2.2)?

The nearest one can get to a covariant spin operator is the set of Pauli–Lubanski operators $\hat{W}_\mu$, defined as follows:

$$\hat{W}_\sigma = -\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\hat{M}^{\mu\nu}\hat{P}^\rho \tag{1.2.3}$$

(with $\epsilon_{0123} = +1$), whose commutation relations can be shown to be

$$\left[\hat{W}_\lambda, \hat{W}_\mu\right] = i\epsilon_{\lambda\mu\rho\sigma}\hat{W}^\rho\hat{P}^\sigma. \tag{1.2.4}$$

These are not quite what we hoped for, but we notice that *if* we consider the action of these operators on states of momentum $\mathbf{p} = 0$, i.e. on 'rest' states, then for the space parts of the commutation relations (1.2.4) one will have

$$\left[\hat{W}_j, \hat{W}_k\right] = i\epsilon_{jk\rho 0}\hat{W}^\rho m = -im\epsilon_{jkl}\hat{W}_l. \tag{1.2.5}$$

Thus, for the case $m \neq 0$ the three operators

$$\hat{s}_i = \frac{1}{m}\hat{W}^i \tag{1.2.6}$$

have the commutation relations

$$[\hat{s}_j, \hat{s}_k] = i\epsilon_{jkl}\hat{s}_l \tag{1.2.7}$$

provided they act on the states of particles at rest.

Further, the operator $\hat{W}_\mu \hat{W}^\mu$ is invariant[1] and its eigenvalues, as can be deduced from(1.2.4)–(1.2.7), are of the form $m^2 s(s+1)$ with $s = 0, \frac{1}{2}, 1, \ldots$. It is the number s that is defined as the 'spin' of a particle in a relativistic theory.

In summary, in a relativistic theory a particle is assigned an invariant spin quantum number s. But only when the particle is at rest can one identify a set of spin operators $\hat{s}_i$ and proceed to invoke the usual formalism of non-relativistic quantum mechanics. Indeed from (1.2.3) one sees that when $\hat{W}_\mu$ acts upon a particle at rest it has the form

$$\hat{W}_\sigma = -\frac{m}{2}\epsilon_{\mu\nu 0\sigma}\hat{M}^{\mu\nu}$$

or, from (1.2.1)

$$\hat{W}^i = m\hat{J}_i. \tag{1.2.8}$$

Thus the $\hat{s}_i$ when acting on states at rest are just the $\hat{J}_i$, so that all the rotational properties of non-relativistic spin hold for particles at rest. The possibility that a particle at rest has non-zero total angular momentum has emerged automatically.

For a particle at rest it is convenient to fix a reference frame and then to classify the states of the particle as in the non-relativistic case, i.e. using eigenstates $|ss_z\rangle$ of $\hat{s}^2$ and $\hat{s}_z$. For a particle in motion, however, the labelling of the states is not so clear cut.

The standard approach is to generate states of arbitrary momentum by acting upon the rest states with suitable Lorentz transformations. We shall adopt an equivalent but more physical approach, considering Lorentz transformations in a similar spirit to our discussion of rotations in Section 1.1.

We denote an arbitrary physical Lorentz transformation by l. We continue to denote physical rotations by r, and we denote by l_j, $j = x, y, z$, physical pure Lorentz transformations ('boosts') along the axes. We remind the reader that care must be taken when specifying a *sequence* of operations acting on the reference system. For example, if we first rotate a system S about its Y axis through angle θ (call this frame S') and then boost to a new frame S'' moving with speed v *along the Z-axis of S'*, then we should represent the complete transformation from S to S'' as

$$S \to S'' = l_{z'}(v) r_y(\theta) S\,;$$

it is essential for clarity to use the primed label z' on l. A *pure* Lorentz transformation or boost in an arbitrary direction is denoted by $l(\mathbf{v})$, where

[1] $\hat{W}_\mu \hat{W}^\mu$ and $\hat{P}_\mu \hat{P}^\mu$ are the only invariant operators of the inhomogeneous Lorentz group.

conventionally

$$l(\mathbf{v}) \equiv \left[r^{-1}(\mathbf{v})\right]'' l_{z'}(v) r(\mathbf{v}). \tag{1.2.9}$$

Here $r(\mathbf{v})$ is the rotation about $\mathbf{e}_{(z)} \times \mathbf{v}$ that rotates the Z-axis into the direction of $\mathbf{v}$ and $(r^{-1}(\mathbf{v}))''$ is its inverse, applied to the boosted frame. We shall refer to (1.2.9) as a *canonical* boost.

The reason for calling (1.2.9) a pure boost is clear from Fig. 1.1, which shows (for the case of $\mathbf{v}$ lying in the XZ plane) that the final reference system S''' has its Z-axis at the same angle θ to $\mathbf{v}$ as did OZ of S.

If a 4-vector A is acted upon by a physical Lorentz transformation l then it is transformed to a new vector, which we shall denote by lA or A_l. Its components are related to those of A by

$$(lA)^\mu \equiv A_l^\mu = \Lambda^\mu{}_\nu(l) A^\nu. \tag{1.2.10}$$

When using matrix notation we shall denote by Λ the 4×4 matrix whose elements are $\Lambda^\mu{}_\nu$, μ referring to the row, ν to the column. The column matrix A is defined to have as components the contravariant components A^ν. Thus (1.2.10) reads as

$$A_l = \Lambda(l) A$$

Explicit forms for $\Lambda^\mu{}_\nu$ for a few cases of special importance follow. If l is simply a rotation r, then we have

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix} \tag{1.2.11}$$

where R is the matrix defined in (1.1.4).

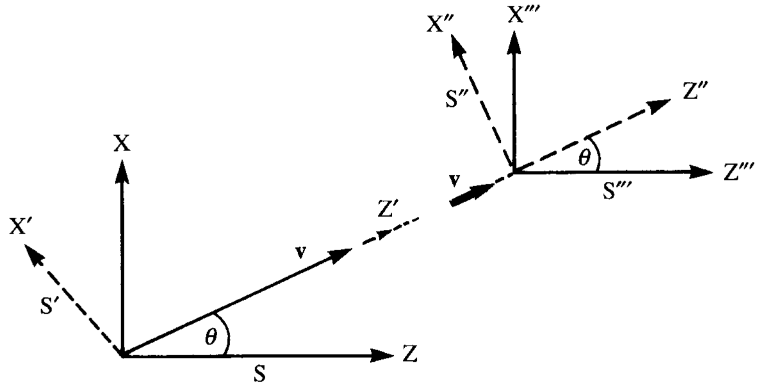

Fig. 1.1. A canonical boost along $\mathbf{v}$ to $S \to S'''$ as shown.