Cambridge University Press & Assessment
978-1-009-39775-9 — Introduction to Electrodynamics
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Excerpt
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1 Vector Analysis

1.1 Vector Algebra

1.1.1 Vector Operations

If you walk 4 miles due north and then 3 miles due east (Fig. 1.1), you will have gone a total of 7 miles, but you're *not* 7 miles from where you set out – only 5. We need an arithmetic to describe quantities like this, which evidently do not add in the ordinary way. The reason they don't, of course, is that **displacements** (straight line segments going from one point to another) have *direction* as well as *magnitude* (length), and it is essential to take both into account when you combine them. Such objects are called **vectors**: velocity, acceleration, force, and momentum are other examples. By contrast, quantities that have magnitude but no direction are called **scalars**: examples include mass, charge, density, and temperature.

I shall use **boldface** (**A**, **B**, and so on) for vectors and ordinary type for scalars. The magnitude of a vector **A** is written |**A**| or, more simply, A. In diagrams, vectors are denoted by arrows: the length of the arrow is proportional to the magnitude of the vector, and the arrowhead indicates its direction. *Minus* **A** (−**A**) is a vector with the same magnitude as **A** but the opposite direction (Fig. 1.2). Note that vectors have magnitude and direction but *not location*: a displacement of 4 miles due north from Washington is represented by the same vector as a displacement 4 miles north from Baltimore (neglecting, of course, the curvature of the Earth). On a diagram, therefore, you can slide the arrow around at will, as long as you don't change its length or direction.

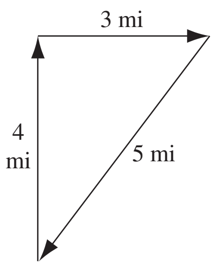

Fig. 1.1

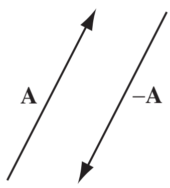

Fig. 1.2

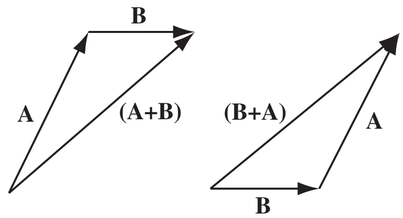

Fig. 1.3

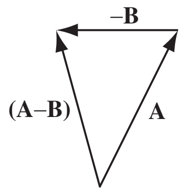

Fig. 1.4

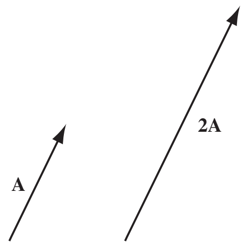

Fig. 1.5

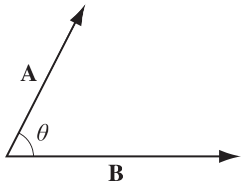

Fig. 1.6

We define four vector operations: addition and three kinds of multiplication.

(i) Addition of two vectors. Place the tail of **B** at the head of **A**; the sum, **A** + **B**, is the vector from the tail of **A** to the head of **B** (Fig. 1.3). (This rule generalizes the obvious procedure for combining two displacements.) Addition is *commutative:*

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A};$$

3 miles east followed by 4 miles north gets you to the same place as 4 miles north followed by 3 miles east. Addition is also *associative:*

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}).$$

To subtract a vector, add its opposite (Fig. 1.4):

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}).$$

(ii) Multiplication by a scalar. Multiplication of a vector by a positive scalar a multiplies the *magnitude* but leaves the direction unchanged (Fig. 1.5). (If a is negative, the direction is reversed.) Scalar multiplication is *distributive:*

$$a(\mathbf{A} + \mathbf{B}) = a\mathbf{A} + a\mathbf{B}.$$

(iii) Dot product of two vectors. The dot product of two vectors is defined by

$$\mathbf{A} \cdot \mathbf{B} \equiv AB\cos\theta, \tag{1.1}$$

where θ is the angle they form when placed tail to tail (Fig. 1.6). Note that $\mathbf{A}\cdot\mathbf{B}$ is itself a *scalar* (hence the alternative name **scalar product**). The dot product is *commutative*,

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A},$$

and *distributive*,

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}. \tag{1.2}$$

Geometrically, $\mathbf{A}\cdot\mathbf{B}$ is the product of A times the projection of $\mathbf{B}$ along $\mathbf{A}$ (or the product of B times the projection of $\mathbf{A}$ along $\mathbf{B}$). If the two vectors are parallel, then $\mathbf{A}\cdot\mathbf{B} = AB$. In particular, for any vector $\mathbf{A}$,

$$\mathbf{A}\cdot\mathbf{A} = A^2. \tag{1.3}$$

If $\mathbf{A}$ and $\mathbf{B}$ are perpendicular, then $\mathbf{A}\cdot\mathbf{B} = 0$.

Example 1.1. Let $\mathbf{C} = \mathbf{A} - \mathbf{B}$ (Fig. 1.7), and calculate the dot product of $\mathbf{C}$ with itself.

Solution.

$$\mathbf{C}\cdot\mathbf{C} = (\mathbf{A}-\mathbf{B})\cdot(\mathbf{A}-\mathbf{B}) = \mathbf{A}\cdot\mathbf{A}-\mathbf{A}\cdot\mathbf{B}-\mathbf{B}\cdot\mathbf{A}+\mathbf{B}\cdot\mathbf{B},$$

or

$$C^2 = A^2 + B^2 - 2AB\cos\theta.$$

This is the **law of cosines**.

(iv) Cross product of two vectors. The cross product of two vectors is defined by

$$\mathbf{A}\times\mathbf{B} \equiv AB\sin\theta\,\hat{\mathbf{n}}, \tag{1.4}$$

where $\hat{\mathbf{n}}$ is a **unit vector** (vector of magnitude 1) pointing perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$. (I shall use a hat ($\hat{}$) to denote unit vectors.) Of course, there are *two* directions perpendicular to any plane: "in" and "out." The ambiguity is resolved by the **right-hand rule**: let your fingers point in the direction of the first vector and curl around (via the smaller angle) toward the second; then your thumb indicates the direction of $\hat{\mathbf{n}}$. (In Fig. 1.8, $\mathbf{A}\times\mathbf{B}$ points *into* the page; $\mathbf{B}\times\mathbf{A}$ points *out* of the page.) Note that $\mathbf{A}\times\mathbf{B}$ is itself a *vector* (hence the alternative name **vector product**). The cross product is *distributive*,

$$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = (\mathbf{A}\times\mathbf{B})+(\mathbf{A}\times\mathbf{C}), \tag{1.5}$$

but *not commutative*. In fact,

$$(\mathbf{B}\times\mathbf{A}) = -(\mathbf{A}\times\mathbf{B}). \tag{1.6}$$

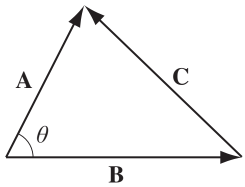

Fig. 1.7

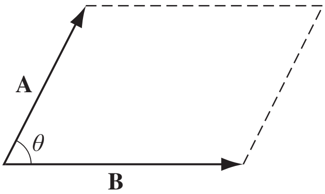

Fig. 1.8

Geometrically, $|\mathbf{A} \times \mathbf{B}|$ is the area of the parallelogram generated by **A** and **B** (Fig. 1.8). If two vectors are parallel, their cross product is zero. In particular,

$$\mathbf{A} \times \mathbf{A} = \mathbf{0}$$

for any vector **A**. (Here **0** is the **zero vector**, with magnitude 0.)

Problem 1.1. Using the definitions in Eqs. 1.1 and 1.4, and appropriate diagrams, show that the dot product and cross product are distributive:

(a) when the three vectors are coplanar;
(b) in the general case.

! **Problem 1.2.** Is the cross product associative?

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \stackrel{?}{=} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}).$$

If so, *prove* it; if not, provide a counterexample (the simpler the better).

1.1.2 Vector Algebra: Component Form

In the previous section, I defined the four vector operations (addition, scalar multiplication, dot product, and cross product) in "abstract" form – that is, without reference to any particular coordinate system. In practice, it is often easier to set up Cartesian coordinates x, y, z and work with vector **components**. Let $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ be unit vectors parallel to the x-, y-, and z-axes, respectively (Fig. 1.9a). An arbitrary vector **A** can be expressed in terms of these **basis vectors** (Fig. 1.9b):

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}.$$

The coefficients A_x, A_y, and A_z are the "components" of **A**; geometrically, they are the projections of **A** along the three coordinate axes ($A_x = \mathbf{A} \cdot \hat{\mathbf{x}}$, $A_y = \mathbf{A} \cdot \hat{\mathbf{y}}$, $A_z = \mathbf{A} \cdot \hat{\mathbf{z}}$). We can now reformulate each of the four vector operations as a rule for manipulating components:

$$\begin{aligned}\mathbf{A} + \mathbf{B} &= (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) + (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}) \\ &= (A_x + B_x)\hat{\mathbf{x}} + (A_y + B_y)\hat{\mathbf{y}} + (A_z + B_z)\hat{\mathbf{z}}.\end{aligned} \quad (1.7)$$

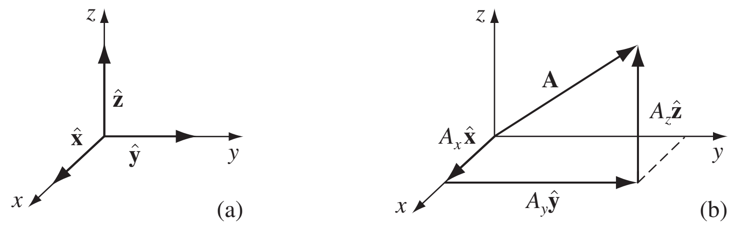

Fig. 1.9

Rule (i) *To add vectors, add like components*:

$$a\mathbf{A} = (aA_x)\hat{\mathbf{x}} + (aA_y)\hat{\mathbf{y}} + (aA_z)\hat{\mathbf{z}}. \tag{1.8}$$

Rule (ii) *To multiply by a scalar, multiply each component.*

Because $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are mutually perpendicular unit vectors,

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1, \quad \hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{x}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = 0. \tag{1.9}$$

Accordingly,

$$\begin{aligned}\mathbf{A} \cdot \mathbf{B} &= (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) \cdot (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}) \\ &= A_xB_x + A_yB_y + A_zB_z. \end{aligned} \tag{1.10}$$

Rule (iii) *To calculate the dot product, multiply like components, and add.* In particular,

$$\mathbf{A} \cdot \mathbf{A} = A_x^2 + A_y^2 + A_z^2,$$

so

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}. \tag{1.11}$$

(This is, if you like, the three-dimensional generalization of the Pythagorean theorem.)

Similarly,[1]

$$\begin{aligned} \hat{\mathbf{x}} \times \hat{\mathbf{x}} &= \hat{\mathbf{y}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = \mathbf{0}, \\ \hat{\mathbf{x}} \times \hat{\mathbf{y}} &= -\hat{\mathbf{y}} \times \hat{\mathbf{x}} = \hat{\mathbf{z}}, \\ \hat{\mathbf{y}} \times \hat{\mathbf{z}} &= -\hat{\mathbf{z}} \times \hat{\mathbf{y}} = \hat{\mathbf{x}}, \\ \hat{\mathbf{z}} \times \hat{\mathbf{x}} &= -\hat{\mathbf{x}} \times \hat{\mathbf{z}} = \hat{\mathbf{y}}. \end{aligned} \tag{1.12}$$

Therefore,

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) \times (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}) \\ &= (A_yB_z - A_zB_y)\hat{\mathbf{x}} + (A_zB_x - A_xB_z)\hat{\mathbf{y}} + (A_xB_y - A_yB_x)\hat{\mathbf{z}}. \end{aligned} \tag{1.13}$$

[1] These signs pertain to a *right-handed* coordinate system (x-axis out of the page, y-axis to the right, z-axis up, or any rotated version thereof). In a *left-handed* system (z-axis down), the signs would be reversed: $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = -\hat{\mathbf{z}}$, and so on. We shall use right-handed systems exclusively. Always check that you haven't inadvertently adopted a left-handed system.

This cumbersome expression can be written more neatly as a determinant:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \tag{1.14}$$

Rule (iv) *To calculate the cross product, form the determinant whose first row is* $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$, *whose second row is* $\mathbf{A}$ *(in component form), and whose third row is* $\mathbf{B}$.

Example 1.2. Find the angle between the face diagonals of a cube.

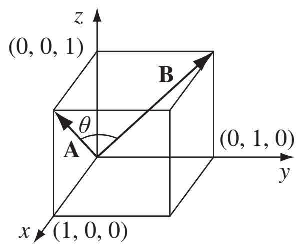

Fig. 1.10

Solution. We might as well use a cube of side 1, and place it as shown in Fig. 1.10, with one corner at the origin. The face diagonals $\mathbf{A}$ and $\mathbf{B}$ are

$$\mathbf{A} = 1\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 1\hat{\mathbf{z}}, \quad \mathbf{B} = 0\hat{\mathbf{x}} + 1\hat{\mathbf{y}} + 1\hat{\mathbf{z}}.$$

So, in component form,

$$\mathbf{A} \cdot \mathbf{B} = 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 1 = 1.$$

On the other hand, in "abstract" form,

$$\mathbf{A} \cdot \mathbf{B} = AB \cos\theta = \sqrt{2}\sqrt{2}\cos\theta = 2\cos\theta.$$

Therefore,

$$\cos\theta = 1/2, \quad \text{or} \quad \theta = 60^\circ.$$

Of course, you can get the answer more easily by drawing in a diagonal across the top of the cube, completing the equilateral triangle. But in cases where the geometry is not so simple, this device of comparing the abstract and component forms of the dot product can be a very efficient means of finding angles.

Problem 1.3. Find the angle between the body diagonals of a cube.

Problem 1.4. Use the cross product to find the components of the unit vector $\hat{\mathbf{n}}$ perpendicular to the shaded plane in Fig. 1.11.

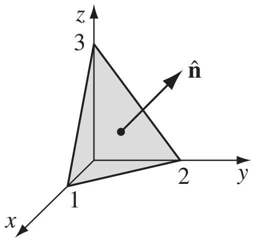

Fig. 1.11

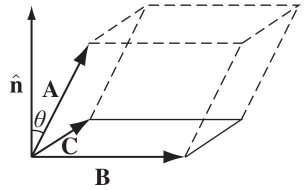

Fig. 1.12

1.1.3 Triple Products

Since the cross product of two vectors is itself a vector, it can be dotted or crossed with a third vector to form a *triple* product.

(i) Scalar triple product: $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$. Geometrically, $|\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})|$ is the volume of the parallelepiped generated by **A**, **B**, and **C**, since $|\mathbf{B}\times\mathbf{C}|$ is the area of the base, and $|\mathbf{A}\cos\theta|$ is the altitude (Fig. 1.12). Evidently,

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}), \tag{1.15}$$

for they all correspond to the same figure. Note that "alphabetical" order is preserved – in view of Eq. 1.6, the "nonalphabetical" triple products

$$\mathbf{A}\cdot(\mathbf{C}\times\mathbf{B}) = \mathbf{B}\cdot(\mathbf{A}\times\mathbf{C}) = \mathbf{C}\cdot(\mathbf{B}\times\mathbf{A}),$$

have the opposite sign. In component form,

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}. \tag{1.16}$$

Note that the dot and cross can be interchanged:

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$$

(this follows immediately from Eq. 1.15); however, the placement of the parentheses is critical: $(\mathbf{A}\cdot\mathbf{B})\times\mathbf{C}$ is a meaningless expression – you can't make a cross product from a *scalar* and a vector.

(ii) Vector triple product: $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$. The vector triple product can be simplified by the so-called **BAC–CAB rule**:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}). \tag{1.17}$$

Notice that

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = -\mathbf{C}\times(\mathbf{A}\times\mathbf{B}) = -\mathbf{A}(\mathbf{B}\cdot\mathbf{C}) + \mathbf{B}(\mathbf{A}\cdot\mathbf{C})$$

is an entirely different vector (cross products are not associative). All *higher* vector products can be similarly reduced, by repeated application of Eq. 1.17, so it is never necessary for an expression to contain more than one cross product in any term. For instance,

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) &= (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}),\\ \mathbf{A}\times[\mathbf{B}\times(\mathbf{C}\times\mathbf{D})] &= \mathbf{B}[\mathbf{A}\cdot(\mathbf{C}\times\mathbf{D})] - (\mathbf{A}\cdot\mathbf{B})(\mathbf{C}\times\mathbf{D}).\end{aligned} \tag{1.18}$$

Problem 1.5. Prove the **BAC–CAB** rule by writing out both sides in component form.

Problem 1.6. Prove that

$$[\mathbf{A}\times(\mathbf{B}\times\mathbf{C})] + [\mathbf{B}\times(\mathbf{C}\times\mathbf{A})] + [\mathbf{C}\times(\mathbf{A}\times\mathbf{B})] = \mathbf{0}.$$

Under what conditions does $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\times\mathbf{C}$?

1.1.4 Position, Displacement, and Separation Vectors

The location of a point in three dimensions can be described by listing its Cartesian coordinates (x, y, z). The vector to that point from the origin ($\mathcal{O}$) is called the **position vector** (Fig. 1.13):

$$\mathbf{r} \equiv x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}. \tag{1.19}$$

I will reserve the letter $\mathbf{r}$ for this purpose, throughout the book. Its magnitude,

$$r = \sqrt{x^2 + y^2 + z^2}, \tag{1.20}$$

is the distance from the origin, and

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}} \tag{1.21}$$

is a unit vector pointing radially outward. The **infinitesimal displacement vector**, from (x, y, z) to $(x + dx, y + dy, z + dz)$, is

$$d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}. \tag{1.22}$$

(We could call this $d\mathbf{r}$, since that's what it *is*, but it's useful to have a special notation for infinitesimal displacements.)

In electrodynamics, one frequently encounters problems involving *two* points – typically, a **source point**, $\mathbf{r}'$, where an electric charge is located, and a **field point**, $\mathbf{r}$, at which you are calculating the electric or magnetic field (Fig. 1.14). It pays to adopt right from the start some shorthand notation for the **separation vector** from the source point to the field point. I shall use for this purpose the cursive letter $\boldsymbol{\mathscr{r}}$:

$$\boldsymbol{\mathscr{r}} \equiv \mathbf{r} - \mathbf{r}'. \tag{1.23}$$

Its magnitude is

$$\mathscr{r} = |\mathbf{r} - \mathbf{r}'|, \tag{1.24}$$

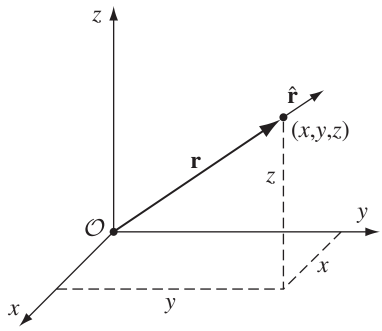

Fig. 1.13

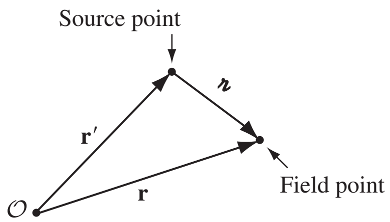

Fig. 1.14

and a unit vector in the direction from $\mathbf{r}'$ to $\mathbf{r}$ is

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}} = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}. \tag{1.25}$$

In Cartesian coordinates,

$$\boldsymbol{\mathscr{r}} = (x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}, \tag{1.26}$$

$$\mathscr{r} = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}, \tag{1.27}$$

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{(x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} \tag{1.28}$$

(from which you will appreciate the economy of the script-$\mathscr{r}$ notation).

Problem 1.7. Find the separation vector $\boldsymbol{\mathscr{r}}$ from the source point $(2, 8, 7)$ to the field point $(4, 6, 8)$. Determine its magnitude ($\mathscr{r}$), and construct the unit vector $\hat{\boldsymbol{\mathscr{r}}}$.

1.1.5 How Vectors Transform

The[2] definition of a vector as "a quantity with a magnitude and direction" is not altogether satisfactory: What precisely does "direction" *mean*? This may seem a pedantic question, but we shall soon encounter a species of derivative that *looks* rather like a vector, and we'll want to know for sure whether it *is* one.

You might be inclined to say that a vector is anything that has three components that combine properly under addition. Well, how about this: We have a barrel of fruit that contains N_x pears, N_y apples, and N_z bananas. Is $\mathbf{N} = N_x\hat{\mathbf{x}} + N_y\hat{\mathbf{y}} + N_z\hat{\mathbf{z}}$ a vector? It has three components, and when you add another barrel with M_x pears, M_y apples, and M_z bananas the result is

[2] This section can be skipped without loss of continuity.

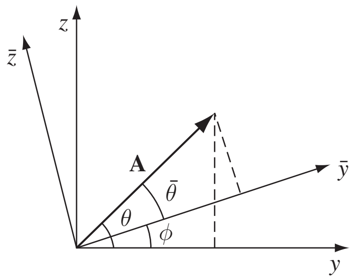

Fig. 1.15

$(N_x + M_x)$ pears, $(N_y + M_y)$ apples, $(N_z + M_z)$ bananas. So it does *add* like a vector. Yet it's obviously *not* a vector, in the physicist's sense of the word, because it doesn't really have a *direction*. What exactly is wrong with it?

The answer is that **N** *does not transform properly when you change coordinates*. The coordinate frame we use to describe positions in space is of course entirely arbitrary, but there is a specific geometrical transformation law for converting vector components from one frame to another. Suppose, for instance, the $\bar{x}, \bar{y}, \bar{z}$ system is rotated by angle ϕ, relative to x, y, z, about the common $x = \bar{x}$-axes. From Fig. 1.15,

$$A_y = A\cos\theta, \quad A_z = A\sin\theta,$$

while

$$\begin{aligned}\bar{A}_y = A\cos\bar{\theta} = A\cos(\theta-\phi) &= A(\cos\theta\cos\phi + \sin\theta\sin\phi)\\ &= \cos\phi\, A_y + \sin\phi\, A_z,\\ \bar{A}_z = A\sin\bar{\theta} = A\sin(\theta-\phi) &= A(\sin\theta\cos\phi - \cos\theta\sin\phi)\\ &= -\sin\phi\, A_y + \cos\phi\, A_z.\end{aligned}$$

We might express this conclusion in matrix notation:

$$\begin{pmatrix}\bar{A}_y\\ \bar{A}_z\end{pmatrix} = \begin{pmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}A_y\\ A_z\end{pmatrix}. \tag{1.29}$$

More generally, for rotation about an *arbitrary* axis in three dimensions, the transformation law takes the form

$$\begin{pmatrix}\bar{A}_x\\ \bar{A}_y\\ \bar{A}_z\end{pmatrix} = \begin{pmatrix}R_{xx} & R_{xy} & R_{xz}\\ R_{yx} & R_{yy} & R_{yz}\\ R_{zx} & R_{zy} & R_{zz}\end{pmatrix}\begin{pmatrix}A_x\\ A_y\\ A_z\end{pmatrix}, \tag{1.30}$$

or, more compactly,

$$\bar{A}_i = \sum_{j=1}^{3} R_{ij}A_j, \tag{1.31}$$

where the index 1 stands for x, 2 for y, and 3 for z. The elements of the matrix R can be ascertained, for a given rotation, by the same sort of trigonometric arguments as we used for a rotation about the x-axis.