An Introduction to Gravity

Einstein's theory of gravity can be difficult to introduce at the undergraduate level, or for self-study. One way to ease its introduction is to construct intermediate theories between the previous successful theory of gravity, Newton's, and our modern theory, Einstein's general relativity. This textbook bridges the gap by merging Newtonian gravity and special relativity (by analogy with electricity and magnetism), a process that both builds intuition about general relativity, and indicates why it has the form that it does. This approach is used to motivate the structure of the full theory, as a nonlinear field equation governing a second-rank tensor with geometric interpretation, and to understand its predictions by comparing it with the – often qualitatively correct – predictions of intermediate theories between Newton's and Einstein's. Suitable for a one-semester course at junior or senior level, this student-friendly approach builds on familiar undergraduate physics to illuminate the structure of general relativity.

Joel Franklin is a professor in the Physics Department of Reed College, Oregon. His research focuses on mathematical and computational methods with applications to classical mechanics, quantum mechanics, electrodynamics, general relativity, and its modifications. He is also the author of textbooks on *Advanced Mechanics and General Relativity*, *Computational Methods for Physics*, *Classical Field Theory*, and *Mathematical Methods for Oscillations and Waves*, all published by Cambridge University Press.

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Preface

Einstein's theory of gravity, general relativity, is over a century old, as established and tested as any fundamental physical theory. It has been around long enough for teachers to develop pedagogically sound approaches to its introduction, at both the undergraduate and graduate level. There are many excellent books on the subject, among my favorites (and ones that informed much of this book) are by Schutz [43], Hartle [23], d'Inverno [10], Wald [51], and the encyclopedic classic by Misner, Thorne, and Wheeler [35]. I recommend these to your attention as primary resources for learning the subject, and delightful references for review and reflection. What, then, is the point of the current offering?

It is striking that gravity, as an undergraduate course, is not taught with the same frequency as, say, electricity and magnetism (E&M). The two are similar in that they deal with a single, specific interaction, making them different from subjects like classical and quantum mechanics, or solid-state physics, that span multiple physical interactions, and which involve a menagerie of techniques for solving their fundamental equations. A course on E&M must also introduce students to techniques for solving its relevant equations, but the relatively narrower set of physical ideas, tightly focused around a single force, can be an advantage in the classroom. Similarly, gravity is a subject with only one interaction (not a force, as it turns out), and as with E&M, there are really only two fundamental objects of study: the field equation that connects the sources to the field, and an equation of motion that describes how particles move in response to that field. Many of the techniques used to teach E&M can be usefully applied to gravity. Even the side benefits of a subject like E&M, like the field-theoretic intuition that is built up naturally as one learns the subject, is shared by an introduction to gravity. So why is gravity set aside in the undergraduate curriculum?

Attempting to answer this question was the starting point for this book. One reason that gravity is difficult to teach is that the field equation, Einstein's equation, involves mathematical objects that are not as familiar as the vector calculus required to learn E&M (and which benefits from the physical intuition that E&M provides). Another is that the interpretation of the field in gravity, the metric that tells us how distance is measured, makes it difficult to understand the observable predictions of the theory. We are used to thinking about a fixed, Minkowski spacetime with forces that act on particles, causing them to move. In general relativity, there is no force – particles move along length-minimizing curves (geodesics) in a spacetime that is determined by the metric, a solution of the field equation given some source configuration. That metric can be expressed in any coordinate system, and figuring out the physical content of its geodesics in an unfamiliar set of coordinates, and in a spacetime that is necessarily very different from Minkowski spacetime, is much harder than using the Lorentz force in Newton's second law to predict motion given an electric and magnetic field.

To address these challenges, I decided to use the question of why Newtonian gravity is "not enough," to both highlight gravitational physics beyond Newtonian gravity (which can be used to introduce some of the more exotic features of general relativity in a familiar setting), and serve as a motivator for the full theory (a motivator both in the sense of why general relativity has to be the way it is, and in the sense of generating student interest, so that they see the value in learning enough geometry to appreciate the subject). Newtonian gravity is virtually identical to electrostatics, and students are very familiar with Newtonian gravity's interpretation and predictions: Mass generates a gravitational force that acts on other masses. It is natural to wonder if the similarity between Newtonian gravity and electrostatics extends to dynamics – does gravity share the relativistic structure of the full set of Maxwell's equations? The question provides a target for discussion, and the setup work involved in answering it is relevant in understanding the linearized limit of general relativity, itself important in building physical intuition about the full theory. There are also differences between gravity and E&M, notably the mass sourcing in gravity, and the opportunity for more general forms of energy to act as a source. Focusing on the similarities and differences between the two theories gives students a framework for thinking about general relativity and its predictions, prior to encountering the less familiar geometric language in which the story of gravity is told. My goal was to create a relatively coherent narrative, starting from the familiar and working incrementally by asking physically motivated questions about the merger of gravity and relativity until the full theory emerges almost as a matter of course. This approach is meant to ease the difficulties with both the formal statement of the theory, and the understanding of its observable predictions by providing a set of familiar footholds along the way to which students can return and regroup as they move along.

Outline

I'll give a brief outline of the book, which will also serve as an informal example syllabus for the course that I teach at Reed College.[1]

[1] Reed College offers a general relativity course every other year. When I teach that course, my audience is junior and senior students who have had a semester of electricity and magnetism at the level of Griffiths [21], and seen special relativity in a sophomore "modern physics" course.

The fundamental question driving the course is "Why can't you take Newtonian gravity, add special relativity, and get a viable relativistic theory of gravity?" So the first thing to do is review Newtonian gravity, and special relativity (both of which my students have seen prior to the class), and begin the process of combining the two, pointing out new predictions along the way. A side effect of this careful attempt to marry gravity and special relativity is that almost everything that shows up in the full theory: black holes, bending of light, even gravitational radiation, is in place, qualitatively, before any really new ideas are introduced. Chapter 1 represents that discussion and is used to build intuition about the subject, while also exploring the deficiencies of the resulting, incomplete theory.

Hopefully, the very physically concrete and accessible Chapter 1 provides motivation for the tool-building Chapter 2. The latter is essentially a review of vector calculus (similar to the review that introductions to E&M provide) set in a new notation. Here I develop the minimal set of machinery, in a flat spacetime setting, that will be needed to present Einstein's equation. Students learn about tensors, their transformation, and are introduced to the metric as a length-measuring device. The focus is on tensors as building blocks of physical theories that are "generally covariant," the same in all coordinate systems. Since those equations typically involve derivatives, and partial derivatives do not always have tensor behavior, I introduce the covariant derivative, together with connections, and the rest of the pieces necessary to discuss geodesics and parallel transport, all in the familiar, Minkowski, setting.

By the end of Chapter 2 (and together with the physics content of Chapter 1), we are led, quite naturally, to a target theory of gravity that is nonlinear, with a second-rank tensor as its field, one that has an almost immediate geometric interpretation. That target theory is then developed in Chapter 3, with only the Riemann tensor needed to set the scene. I present the standard "derivation" of Einstein's equation by comparison of the trajectories of nearby falling masses with geodesic deviation. My hope is that at this point, there is a sense of inevitability that drives students along. Before solving Einstein's field equation, I make contact with the incomplete "gravito-electo-magnetic" theory from Chapter 1 by working through the linearized limit of both the field equation and geodesic equation of motion. This is meant to give students a way to understand the physics predicted by the full theory. The linearized limit of gravity is also a nice one for comparison with E&M, and I discuss its solutions in that context.

Chapter 4 explores the Schwarzschild solution. I make use of symbolic computational packages to ease the calculation of the Ricci tensor, so that we can focus instead on the content of the field equation, and its solution. With this first nontrivial vacuum solution in place, I go through the usual set of geodesic observations, coordinate transformations, and experimental tests, noting the qualitative similarity to the predictions made in Chapter 1 while also highlighting the quantitative differences. There are both linearized geodesic calculations, and numerical ones, and students are invited to explore geodesics in spacetimes like Kerr using these techniques.

Staying in vacuum, plane waves and radiation are the topics in Chapter 5. I work only in the linearized limit of general relativity, and so I rely heavily on E&M as an example theory. Almost all of the development of gravitational radiation is done in parallel with descriptions of electromagnetic radiation. This is true in building vacuum solutions from (far away) sources using Green's functions and limits that define radiation. But the parallel approach is also useful in thinking about the detection of radiation by looking at the particle motion resulting from interaction with the field. Comparison and contrast between gravity and E&M are the main pedagogical tactics.

Moving from vacuum to "material" in the Chapter 6, I discuss continuum mechanics in both Newtonian and relativistic settings. Descriptions of the stress tensor sources for gravity, and their conservation, benefit from a review of electromagnetic sources. As topics, stellar interiors and a brief introduction to cosmology form the bulk of the applications, but I take the "in material" opportunity to present wormholes and the Alcubierre warp drive, both of which require (exotic) sources. These last topics represent a nice inversion of Einstein's equation, in which a metric with desired properties is built, and the stress tensor required to generate that metric is then characterized (as opposed to starting with a source and finding the field, the usual direction).

Chapter 7 is somewhat indulgent given my background and interests, and could easily be omitted – my ultimate goal is to round out the general relativistic solutions that were foreshadowed in Chapter 1 by solving Einstein's equation for a spherically symmetric central mass that also contains charge. That source is extended, like the ones in Chapter 6, but rather than study it in that setting, I tried to add a little value by introducing field actions for scalar and vector fields. Then adding in the Einstein–Hilbert action allows for a discussion of universal coupling and a concrete place to think about the role of the metric in scalar/vector field equations together with the role of scalar/vector field sources in Einstein's equation. Once we see how any theory necessarily couples to gravity, I develop the Reissner–Nordstrøm solution, as a final spacetime for study.

There are three appendices that I use only for reference when I teach the course. Appendix A is a bare bones review of special relativity, which, again, my students have seen (usually multiple times). For completeness, I include Appendix B on the Runge–Kutta method, which can be used to solve for geodesic trajectories in any spacetime of interest. While there are numerical differential equation solvers built in to many programming language libraries by now, it's always a good idea to understand what they are doing to avoid identifying a numerical artifact as a physical feature. It is also the case that the fourth-order Runge–Kutta method is widely used and requires only about ten lines of code to implement, making it worth the minimal effort. Lastly, while the curvature that is relevant in general relativity is somewhat different from the curvature of curves and surfaces, there is some intuition that can be built by thinking about the latter, and so I have a short introduction to those curvature ideas as Appendix C.

Notes

That overview completed, I'll note some personal idiosyncrasies to help avoid distraction when reading the text and carrying out the exercises:

1. Notation: I use fairly standard tensor notation, and have tried to highlight those places where I stray, temporarily, for the sake of clarity. In general, Greek indices occur when sums are over the full spacetime or when I do not need to distinguish between time and space components. Roman indices are used to refer to the spatial components of a tensor that has both time and space components. When dealing with vectors that have components and basis in place, I use bold typeface, for example, $\mathbf{E} = E^x \hat{\mathbf{x}} + E^y \hat{\mathbf{y}} + E^z \hat{\mathbf{z}}$. I use blackboard bold for matrices (or, if focusing on a matrix-like table representing a tensor, I'll use an indexed object). When writing out a matrix as part of an assignment or equality, I use the notation $\doteq$ rather than $=$, a reminder that these are representations of an object rather than the object itself, a habit picked up from Sakurai's quantum mechanics text (J. J. Sakurai, *Modern Quantum Mechanics Revised Edition*, Addison-Wesley, 1994).

   General relativity is a subject that contains a lot of symbolic information, with tensor indices dangling here and there, and other reminders cluttering expressions. One place where some economy can be found is in indicating functional dependence. When you have a tensor $T^\mu(t, x, y, z)$ that depends on the coordinates, I tend to write $T^\mu(x)$ with x standing in for all the coordinates. If I want to highlight a spacetime split in functional dependence, I use $T^\mu(\mathbf{r}, t)$ where $\mathbf{r}$ represents the spatial pieces (sometimes I use $\mathbf{x}$ to do the same thing, depending on what other symbols are nearby). There are times when indicating the dependence is unwieldy, as in expressions like $S[A^\mu(x)]$, the "action depending on the field A^μ, itself a function of the coordinates," and I usually omit the innermost set of dependence reminders. When the functional dependence is not the point, I omit it altogether, so you'll see things like $\dot{x}^\alpha$ standing in for $\dot{x}^\alpha(t)$.
2. References: I have tried to include original sources where appropriate, together with some interesting papers specifically aimed at undergraduate physics majors. Beyond that, and as a general rule, most of the topics in this book can be found in the books on gravity listed above (and many others), all of which have much more extensive references for advanced study.
3. Exercises: To quote from Mary Boas's excellent *Mathematical Methods in the Physical Sciences* (3rd ed. John Wiley & Sons, 2006): (replace "mathematics" with "physics" in the current setting) "To use mathematics effectively in applications, you need not just knowledge but *skill*. Skill can be obtained only through practice... The *only* way to develop the skill necessary to use this material in your later courses is to practice by solving many problems. Always study with pencil and paper at hand. Don't just

read through a solved problem – try to do it yourself! Then solve similar ones from the problem set for that section . . . ” Sage advice, and while I have not included problems at the end of sections, I do reference problems within the text itself. You should work those problems when you run across them in your reading, and do the rest when you get to the end of the chapter.

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I learned general relativity from Professor Stanley Deser, who was a wonderful mentor to me. Much of the approach used in this book is informed by my interactions with him. He impressed upon me the utility of thinking about difficult physics by analogy with simpler physics as a way to make progress. I have attempted to do that (and using one of his favorite vehicles for comparison, E&M) in this book and hope that he would have enjoyed it. My postdoctoral adviser Professor Scott Hughes taught me the importance of making clear and concrete physical predictions in general relativity. That has never been easy (for me) to do, but it sure is fun when you can pull it off, and well worth the effort of trying. Professor David Griffiths remains the clearest and most creative expositor of physics that I have ever encountered, producing an infuriating standard that is, nevertheless, something to shoot for. He provided commentary and suggestions for this text, for which I am extremely grateful. Of course, it would be even better if he would just write a book on gravity . . . Finally, I'd like to thank the Physics Department at Reed College – my colleagues and our students in the department are an inspiration.