1 Newtonian Gravity

Newton's law of universal gravitation was the accepted theory of gravity until general relativity supplanted it. Because of its predictive power in a nonrelativistic limit, Newtonian gravity retains an important place in the physics curriculum. It provides quantitative accuracy for many cases of interest. One of its first feats was providing a theoretical description of Kepler's (observational) laws. Yet there is a problem with this simplified form of gravity: Its description relies on the concept of force. This is a problem because the modern theory of gravity, while it reduces to a theory that looks like Newtonian gravity in approximation, is fundamentally different. General relativity provides a force-free gravitational interaction. It may not appear to matter much, but a reliance on Newtonian gravity pollutes our description of physics almost from the very start. We talk about the "four forces of nature," a simple and lovely idea that suggests that all interparticle relationships can be boiled down to four types. This way of thinking, in part, also invites reduction. The electromagnetic and weak forces, for example, can be combined into one. Can further simplification be found? Is it possible that all of these forces are manifestations of a single interaction, just viewed through different lenses, at different energy scales?

No. At least, not without an intermediate idea that would allow the electroweak and strong forces to be combined with gravity, which can be viewed as a geometric effect. That shift from force to geometry is not an incremental one that can be easily bridged. One focus of this chapter is to establish the structure of the gravitational field. By thinking about bringing Newtonian gravity in line with special relativity we can see the structural elements that are required. In particular, we will show that a theory of gravity (1) must be nonlinear, (2) must involve a second-rank tensor (as both source and field), and then in Chapter 2, we will finish the job by showing that (3) only the symmetric part of that tensor is physically relevant, and at that point, one can view the theory as a geometric one. So we will see that the target theory, which will end up being general relativity, is highly constrained from the start by special relativity. I want to highlight this shift away from gravity as a force as much as possible, especially since the "gravitational force" is so ingrained in our collective educational toolbox. But, to develop the implications of special relativity for a gravitational interaction, it will be easiest to work from Newtonian gravity, modifying it and adding to it as we consider more and more implications of special relativity. That morphing will occur very much within a force framework, and much of it will be done by comparison with electricity and magnetism, another force-based theory of interaction. Consider yourself warned, then, that while we will use familiar descriptions of gravity, our end point in this chapter and the next are that those familiar descriptions cannot hold.

The chapter begins with a description of Newtonian gravity as a field theory, meaning that we will be focused on the divergence and curl of the gravitational field, relating it to its sources and characterizing its vector geometry. Much of this will be a review, both of vector calculus and of the usual observations about a gravitational field, like the orbital motion of massive bodies in the presence of the field, the existence of a gravitational potential, analogous to an electrostatic one, and the energy required to build configurations of mass, energy that is then stored in the field itself, and so on.

After this review, we'll think about the implications of adding "a little" special relativity. In particular, what happens if we use $E = mc^2$ to relate the massive sources that we know from Newtonian gravity to energetic sources? This shift, from mass to energy, will also allow us to consider the response of particles that have energy but no mass to gravitational fields. This opens up a number of interesting possibilities. We can consider the bending of light by a gravitational field and also the gravitational field that is generated by a beam of light. We can also compute the gravitational contribution from static electric fields and find the gravitational field that comes from a massive point charge. Just as electric fields have an associated energy density, gravitational fields also carry energy and therefore must act as sources for themselves. That idea already establishes that a theory of gravity must be nonlinear, as the self-coupling energy density term goes like the field-squared, just as it does for an electric or magnetic field.

We'll make further progress towards our goal by considering the gravitational field in different inertial frames related by a Lorentz boost. Just as one can show that a magnetic field must exist by demanding that the predictions of two frames match a manifestation of "the laws of physics are the same in all inertial frames," the tenet of special relativity, you can also establish that there must be a "gravitomagnetic" field that is sourced by moving mass. Once the gravitomagnetic field is in place, we get "the rest" of the "Maxwell" equations easily. And so, without appealing to general relativity at all, we will see that, for example, gravitational radiation is already present as a theoretical prediction with even a naive marriage of Newtonian gravity and special relativity. Many of the extensions that we discuss in this chapter are qualitatively correct in full GR, often differing only by factors of two and four. But please don't forget that the ultimate structure of general relativity is very different from these force-field observations.

1.1 Two Observations and Their Consequences

All of Newtonian gravity can be built from two fundamental observations. The first is the experimentally verifiable form of the force between two masses m_1 and m_2. For m_1 at vector location $\mathbf{r}_1$ and m_2 at location $\mathbf{r}_2$ as shown in Figure 1.1, the force on the first mass due to the second is

$$\mathbf{F}_{12} = -\frac{G m_1 m_2}{R_{12}^2} \hat{\mathbf{R}}_{12}, \tag{1.1}$$

where $G \approx 6.67 \times 10^{-11}\,\mathrm{N\,m^2/kg^2}$ is the gravitational constant. The force has magnitude proportional to the product of the masses and acts along the line connecting the centers of mass of each. The minus sign out front tells us that the force is attractive, tending to pull the first mass towards the second. We have defined the vector $\mathbf{R}_{12} = \mathbf{r}_1 - \mathbf{r}_2$ to be the vector pointing from $\mathbf{r}_2$ to $\mathbf{r}_1$, with associated unit vector $\hat{\mathbf{R}}_{12}$. The force acting on mass two due to mass one is just the negative of the force acting on mass one due to mass two: $\mathbf{F}_{21} = -\mathbf{F}_{12}$.

Suppose you have several masses $\{m_i\}_{i=1}^N$ at locations $\{\mathbf{r}_i\}_{i=1}^N$. A second observation about the gravitational force is that the net force acting on a mass m at location $\mathbf{r}$ is the sum of the individual forces from each of the N masses. Referring to Figure 1.2, the total force on m is

$$\mathbf{F} = -\sum_{i=1}^N \frac{G m m_i}{R_i^2} \hat{\mathbf{R}}_i, \qquad \mathbf{R}_i \equiv \mathbf{r} - \mathbf{r}_i, \tag{1.2}$$

where the individual separation vectors are $\mathbf{R}_i \equiv \mathbf{r} - \mathbf{r}_i$, pointing from the ith mass to m, and $\hat{\mathbf{R}}_i$ is the associated unit vector. This idea, that the net force is the vector sum of each mass's force contribution, is known as "superposi-

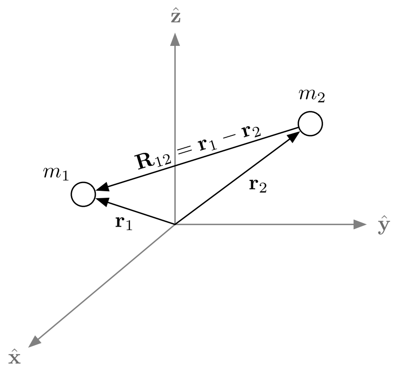

Fig. 1.1 Mass m_1 is located at $\mathbf{r}_1$, mass m_2 at $\mathbf{r}_2$. The vector $\mathbf{R}_{12} = \mathbf{r}_1 - \mathbf{r}_2$ is the vector that points from $\mathbf{r}_2$ to $\mathbf{r}_1$.

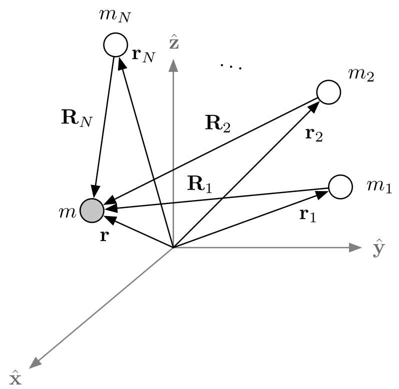

Fig. 1.2 A set of masses $\{m_i\}_{i=1}^N$ at locations $\{\mathbf{r}_i\}_{i=1}^N$ exert gravitational forces on a mass m at $\mathbf{r}$. The observation is that the net force on m is the sum of the individual forces.

tion." It is a familiar concept, but one that we highlight must be established experimentally, superposition is not a given.

In the force expression from (1.2), the mass of the target particle, m, appears in each term of the sum. We can define the gravitational field to be the force per unit mass acting at location $\mathbf{r}$, similar to the electric field's force per unit charge definition. Let $\mathbf{g} = \mathbf{F}/m$, so that for the setup in Figure 1.2, the gravitational field at $\mathbf{r}$ is

$$\mathbf{g}(\mathbf{r}) = -\sum_{i=1}^{N} \frac{Gm_i}{R_i^2}\hat{\mathbf{R}}_i \tag{1.3}$$

with units of acceleration. As in E&M, we will focus on the field. To recover the force, just multiply the field by m: $\mathbf{F} = m\mathbf{g}$. The mass m at $\mathbf{r}$ will play the role of a "test mass" just as we often think of a positive "test charge" when working with the electric and magnetic fields.

We can generalize the field in (1.3) to apply to continuous distributions of mass. If we have a mass density function $\rho(\mathbf{r}')$ that gives the mass per unit volume at the point $\mathbf{r}'$, then in a small volume $d\tau'$ surrounding the point $\mathbf{r}'$, there is mass $dm' = \rho(\mathbf{r}')\,d\tau'$. Referring to Figure 1.3, the contribution of that little patch of mass to the gravitational field at $\mathbf{r}$ is

$$d\mathbf{g} = -\frac{G\,dm'}{R^2}\hat{\mathbf{R}} = -\frac{G\rho(\mathbf{r}')d\tau'}{R^2}\hat{\mathbf{R}}, \qquad \mathbf{R} \equiv \mathbf{r} - \mathbf{r}', \tag{1.4}$$

and then using superposition, we can sum up all those pieces to arrive at a continuous form of (1.3),

$$\mathbf{g}(\mathbf{r}) = \int d\mathbf{g} = -\int \frac{G\rho(\mathbf{r}')}{R^2}\hat{\mathbf{R}}\,d\tau', \tag{1.5}$$

where the volume integral is over all space, with the understanding that $\rho(\mathbf{r}') = 0$ where there is no source mass. In Cartesian coordinates, the volume element is $d\tau' = dx'dy'dz'$; we are integrating over the primed source coordinates.

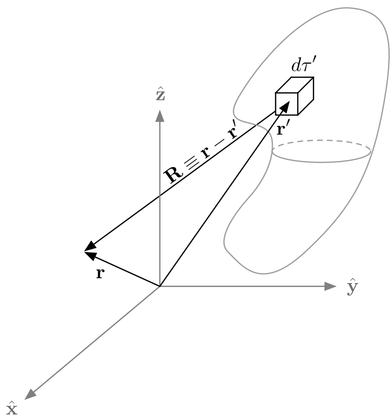

Fig. 1.3 A continuous distribution of mass described by the mass density $\rho(\mathbf{r}')$, which gives the mass per unit volume at $\mathbf{r}'$. The mass in the small $d\tau'$ volume shown is $dm' = \rho(\mathbf{r}')d\tau'$. That mass will contribute to the gravitational field at the point $\mathbf{r}$.

The integral expression for the field in (1.5) can in principle be used to find the gravitational field at any point $\mathbf{r}$ for any given distribution $\rho(\mathbf{r}')$ that is localized (and for which the integral exists). It encapsulates our two observations, the vector form of the field due to a point source and superposition. But it is often useful to have differential equations relating $\mathbf{g}$ to its source ρ. We can obtain these by taking the divergence and curl of $\mathbf{g}(\mathbf{r})$ directly from (1.5). Taking the curl, which slips through the integral (the curl has derivatives with respect to $\mathbf{r}$ in it, whereas the integral is over the dummy variables $\mathbf{r}'$),

$$\nabla \times \mathbf{g}(\mathbf{r}) = -G \int \rho(\mathbf{r}') \nabla \times \left(\frac{\hat{\mathbf{R}}}{R^2} \right) d\tau'. \tag{1.6}$$

We can evaluate the curl of $\hat{\mathbf{R}}/R^2 = \mathbf{R}/R^3$,[1]

$$\nabla \times \frac{\mathbf{R}}{R^3} = \frac{\nabla \times \mathbf{r}}{R^3} + \nabla\left(\frac{1}{R^3}\right) \times \mathbf{r} - \frac{\nabla \times \mathbf{r}'}{R^3} - \nabla\left(\frac{1}{R^3}\right) \times \mathbf{r}', \tag{1.7}$$

and it is clear that $\nabla \times \mathbf{r} = 0$ from the definition of the curl. The curl of $\mathbf{r}'$ is zero because $\mathbf{r}' = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}} + z'\hat{\mathbf{z}}$ doesn't depend on x, y, or z, which is what ∇ acts on. The gradient of R^{-3} is

$$\nabla\left[\left((x-x')^2 + (y-y')^2 + (z-z')^2\right)^{-3/2}\right] = -3\frac{\mathbf{R}}{R^5}. \tag{1.8}$$

Using the relation from (1.8) in (1.7) gives

$$\nabla \times \frac{\mathbf{R}}{R^3} = -3\frac{\mathbf{R} \times \mathbf{r}}{R^5} + 3\frac{\mathbf{R} \times \mathbf{r}'}{R^5} = -3\frac{\mathbf{R} \times \mathbf{R}}{R^5} = 0. \tag{1.9}$$

[1] The relevant product rule for curls acting on a function a and vector function $\mathbf{B}$ is

$$\nabla \times (a\mathbf{B}) = a(\nabla \times \mathbf{B}) + (\nabla a) \times \mathbf{B}.$$

This result makes good geometric sense, since the curl measures the extent to which a vector field "curls" around a point, and the vector field $\mathbf{R}/R^3$ points radially away from the point $\mathbf{r}'$, it doesn't curl around anything. The gravitational field is made up of a sum of these divergent pieces, and (1.9) used in (1.6) makes it is clear that $\nabla \times \mathbf{g}(\mathbf{r}) = 0$.

The divergence of the gravitational field is

$$\nabla \cdot \mathbf{g}(\mathbf{r}) = -G \int \rho(\mathbf{r}')\nabla \cdot \left(\frac{\hat{\mathbf{R}}}{R^2}\right) d\tau'. \tag{1.10}$$

We have to be careful with the divergence inside the integrand here, $\nabla \cdot (\mathbf{R}/R^3)$. If we just calculate it directly,[2]

$$\nabla \cdot \left(\frac{\mathbf{R}}{R^3}\right) = -3\frac{\mathbf{R}}{R^5} \cdot \mathbf{R} + \frac{3}{R^3} = 0. \tag{1.11}$$

This result is surprising since the vector field diverges strongly from the point $\mathbf{r}'$. The calculation in (1.11) is valid at all points except for $\mathbf{r}'$ itself, so the divergence is indeed zero away from the source point. But it is possible that the divergence *at* $\mathbf{r}'$ is nonzero (since at this point we are dividing zero by zero), and we hope that this is the case to save our intuitive expectation. It is not immediately obvious how to probe the point $\mathbf{r}'$ using the derivative operator ∇, but we can remove that derivative by integrating over a ball of radius ϵ centered at the point $\mathbf{r}'$ and using the divergence theorem.

To carry out the integral, move the origin of the coordinate system to $\mathbf{r}'$, so that $\mathbf{R} = \mathbf{r}$, a vector pointing from the origin to any point in three dimensions (after we evaluate the integral, we will move the origin back to $\mathbf{r}'$). For $B(\epsilon, 0)$, the ball of radius ϵ centered at the origin, the divergence theorem gives

$$\int_{B(\epsilon,0)} \nabla \cdot \frac{\hat{\mathbf{r}}}{r^2}\, d\tau = \oint_{\partial B(\epsilon,0)} \frac{\hat{\mathbf{r}}}{r^2} \cdot d\mathbf{a}, \tag{1.12}$$

with $r^2 = \epsilon^2$ at the surface of the ball. That surface has area element $d\mathbf{a} = \epsilon^2 \sin\theta d\theta d\phi \hat{\mathbf{r}}$ in spherical coordinates. Performing the surface integral,

$$\oint_{\partial B(\epsilon,0)} \frac{\hat{\mathbf{r}}}{r^2} \cdot d\mathbf{a} = \int_0^{2\pi} \int_0^{\pi} d\theta d\phi = 4\pi. \tag{1.13}$$

Now we know that $\nabla \cdot (\hat{\mathbf{r}}/r^2)$ is zero everywhere except at the origin, with indeterminate value there, but a value that can be integrated to 4π. That is practically the definition of the Dirac delta function,[3] and we conclude that

[2] Here we use the product rule for function a and vector $\mathbf{B}$,

$$\nabla \cdot (a\mathbf{B}) = \nabla a \cdot \mathbf{B} + a\nabla \cdot \mathbf{B}.$$

[3] The one-dimensional Dirac delta function is defined by its integral behavior: $\delta(x) = 0$ unless $x = 0$ where it is infinite but such that

$$\int_{-\infty}^{\infty} \delta(x)dx = 1.$$

The three-dimensional form is a product: $\delta(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$. Try Problem 1.1 and Problem 1.2 for a review of the Dirac delta function.

$\nabla \cdot (\hat{\mathbf{r}}/r^2) = 4\pi\delta^3(\mathbf{r})$ or, moving the origin back,

$$\nabla \cdot \left(\frac{\hat{\mathbf{R}}}{R^2}\right) = 4\pi\delta^3(\mathbf{r} - \mathbf{r}'). \tag{1.14}$$

Returning to (1.10) with (1.14), we learn that the divergence of the gravitational field is

$$\nabla \cdot \mathbf{g}(\mathbf{r}) = -G \int \rho(\mathbf{r}') 4\pi\delta^3(\mathbf{r} - \mathbf{r}')\, d\tau' = -4\pi G\rho(\mathbf{r}). \tag{1.15}$$

Our original two observations about the gravitational force have become the pair of partial differential equations for the gravitational field:

$$\boxed{\nabla \cdot \mathbf{g}(\mathbf{r}) = -4\pi G\rho(\mathbf{r}), \qquad \nabla \times \mathbf{g}(\mathbf{r}) = 0,} \tag{1.16}$$

providing a force on a particle of mass m at $\mathbf{r}$ of $\mathbf{F} = m\mathbf{g}(\mathbf{r})$. You should compare these equations with the similar ones that govern the electrostatic field

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho_e(\mathbf{r})}{\epsilon_0}, \qquad \nabla \times \mathbf{E}(\mathbf{r}) = 0, \tag{1.17}$$

for $\rho_e(\mathbf{r})$ the charge density. The force on a charge q at $\mathbf{r}$ is $\mathbf{F} = q\mathbf{E}(\mathbf{r})$. The comparison gives us a way to formally map electrostatic results to gravitational ones by taking $\epsilon_0^{-1} \rightarrow -4\pi G$ and $\rho_e \rightarrow \rho$.

1.2 The Field Equations

Both the divergence and curl carry useful information about the field $\mathbf{g}$. If we integrate the equation governing the divergence of $\mathbf{g}$ over an arbitrary volume Ω and use the divergence theorem, we return to an integral form,

$$\int_\Omega \nabla \cdot \mathbf{g}(\mathbf{r})\, d\tau = -4\pi G \int_\Omega \rho(\mathbf{r})\, d\tau \longrightarrow \oint_{\partial\Omega} \mathbf{g}(\mathbf{r}) \cdot d\mathbf{a} = -4\pi G M_{\text{enc}}, \tag{1.18}$$

where M_{enc} is the mass enclosed by the volume. The gravitational field flux through the surface enclosing Ω is proportional to the mass enclosed, much like Gauss's law. We can use the integral form to find the gravitational field due to highly symmetric distributions of mass.

Example 1.1 (Spherical Shell of Mass). What is the gravitational field inside and outside a sphere of radius R with mass M uniformly distributed over its surface? Assume that the sphere is centered at the origin of our coordinate system. Since the source is spherically symmetric, it is reasonable to take the gravitational field to be spherically symmetric[4] which, for a vector field, means that the magnitude should depend only on the distance to the origin, with direction that is radial: $\mathbf{g}(\mathbf{r}) = g(r)\hat{\mathbf{r}}$.

Take a "Gaussian surface" that is a sphere of radius $r < R$. There is no mass enclosed by that sphere, $M_{\text{enc}} = 0$. The surface integral in (1.18) is easy to evaluate from the assumed spherically symmetric form,

$$\oint \mathbf{g}(\mathbf{r}) \cdot d\mathbf{a} = g(r) 4\pi r^2, \tag{1.19}$$

and putting these two pieces together in (1.18), we learn that $g(r)4\pi r^2 = 0$, so that $\mathbf{g}(\mathbf{r}) = 0$ for points inside the spherical shell.

If our Gaussian surface has radius $r > R$, then all that changes is the mass enclosed, $M_{\text{enc}} = M$. Now we have $g(r)4\pi r^2 = -4\pi GM$ giving $\mathbf{g}(r) = -GM/r^2\hat{\mathbf{r}}$. We have recovered the usual "shell theorem" result: Inside the sphere, there is no field, and outside the field is that of a point source of mass M sitting at the origin.

Example 1.2 (Infinite Line of Mass). As another case where we can use this gravitational Gauss's law, take an infinite uniform line of mass lying along the $\hat{\mathbf{z}}$ axis. The mass per unit length of the line is λ, what is the gravitational field a distance s from the line? This time, we will take $\mathbf{g}(\mathbf{r})$ to have cylindrical symmetry, and its magnitude depends only on s, the distance to the axis, with direction radially away from the axis, $\mathbf{g}(\mathbf{r}) = g(s)\hat{\mathbf{s}}$ (where s is the distance to the $\hat{\mathbf{z}}$ axis, and $\hat{\mathbf{s}}$ is its direction of increase). For the Gaussian surface, take a cylinder of radius s and height ℓ centered on the line. The mass enclosed is $M_{\text{enc}} = \lambda\ell$, and the surface integral, easily evaluated in cylindrical coordinates, is

$$\oint \mathbf{g}(\mathbf{r}) \cdot d\mathbf{a} = g(s)\ell 2\pi s. \tag{1.20}$$

When used in (1.18), $g(s)\ell 2\pi s = -4\pi G\lambda\ell \rightarrow \mathbf{g}(\mathbf{r}) = -2G\lambda/s\hat{\mathbf{s}}$.

This is a good place to see how the mapping $\epsilon_0^{-1} \rightarrow -4\pi G$ can be useful in generating gravitational results from electrostatic ones. For an infinite line of electric charge with charge density λ_e, the electric field is $\mathbf{E}(\mathbf{r}) = \lambda_e/(2\pi\epsilon_0 s)\hat{\mathbf{s}}$. We would predict, from the map, that

$$\mathbf{E}(\mathbf{r}) = \frac{\lambda_e}{2\pi\epsilon_0 s}\hat{\mathbf{s}} \longrightarrow \mathbf{g}(\mathbf{r}) = -\frac{2G\lambda}{s}\hat{\mathbf{s}}, \tag{1.21}$$

confirming our result above.

We can keep going, of course, finding the gravitational fields associated with infinite cylinders, sheets and blocks of mass. You should try solving Prob-

[4] Why is that? There is nothing in the equation $\nabla \cdot \mathbf{g} = -4\pi G\rho$ that indicates that the symmetry of the source should be reflected in the field. But a partial differential equation by itself is not enough to find the field. There are also boundary conditions to consider, and these inherit the source symmetries and propagate those symmetries to the field.

lem 1.3, Problem 1.4, and Problem 1.5 using both the gravitational Gauss' law and the electrostatic to gravitational map.

So much for the divergence of the gravitational field and its integral form. How about the curl? Pick a surface S with boundary ∂S and integrate $\nabla \times \mathbf{g} = 0$ over it using the curl theorem to turn the surface integral into a boundary integral

$$0 = \int_S \big(\nabla \times \mathbf{g}(\mathbf{r})\big) \cdot d\mathbf{a} = \oint_{\partial S} \mathbf{g}(\mathbf{r}) \cdot d\boldsymbol{\ell}. \tag{1.22}$$

If we multiply this equation by m, then it says that the work done around a closed loop is zero, the gravitational force is conservative. Then it can be developed from a potential energy function U via the gradient. While this is a familiar statement of conservation, we can work directly with the curl of $\mathbf{g}$ itself. The most general vector field that has curl vanishing everywhere is the gradient of a scalar, so $\nabla \times \mathbf{g}(\mathbf{r}) = 0 \rightarrow \mathbf{g}(\mathbf{r}) = -\nabla\varphi(\mathbf{r})$. The minus sign is traditional (think of $\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})$ from E&M), and since we are dealing with the field, φ is an energy per unit mass (just as V is energy per unit charge) called the "gravitational potential."[5] The potential energy is related to the potential by a factor of m, $U = m\varphi$, just as the field $\mathbf{g}$ is related to the force.

The gravitational potential allows us to combine the content of the pair in (1.16) into a single equation. By construction, $\nabla \times \mathbf{g}(\mathbf{r}) = 0$, so we only have to worry about the divergence, $\nabla \cdot (-\nabla\varphi(\mathbf{r})) = -4\pi G\rho(\mathbf{r})$, or

$$\boxed{\nabla^2\varphi(\mathbf{r}) = 4\pi G\rho(\mathbf{r}),} \tag{1.23}$$

which is Poisson's equation. The linearity of Poisson's equation tells us that superposition holds here: adding sources adds their contribution to the potential. That allows us to develop an integral solution to (1.23) by building up combinations of its point source solution. A point source sitting at the origin has mass density $\rho(\mathbf{r}) = m\delta^3(\mathbf{r})$. How should we think about solving (1.23) for this source? Since the source is spherically symmetric, we expect φ to be as well (again in order to impose boundary conditions), and a function is spherically symmetric if it depends only on the distance to the origin: $\varphi(\mathbf{r}) = \varphi(r)$. Under this assumption, the Laplacian simplifies to $\nabla^2\varphi(\mathbf{r}) = (r\varphi(r))''/r$ with primes denoting r derivatives. If we work away from the source, at points with $r > 0$, then Poisson's equation reads

$$\frac{d^2}{dr^2}\big(r\varphi(r)\big) = 0 \longrightarrow \varphi(r) = \frac{\alpha}{r} + \beta \tag{1.24}$$

for constants of integration α and β. To set those constants, we rely on boundary conditions. One generally implicit one is that the potential should vanish

5 The units of φ are already tantalizing: energy per unit mass is $(\mathrm{m/s})^2$, a speed squared. In Newtonian gravity, there is no natural speed that can be made out of the available constants, G and m, so there is nothing to set a scale for the gravitational potential. If we were working in a special relativistic setting, though, we *would* have a natural speed, the speed of light c.

at spatial infinity, $\varphi(\mathbf{r}) \to 0$ as $r \to \infty$, requiring that $\beta = 0$.[6] The other boundary condition comes from $r = 0$, the source of the delta function in the field. Integrating (1.23) with the delta source over a ball of radius ϵ centered on the mass and again using the divergence theorem on $\nabla^2\varphi = \nabla \cdot (\nabla\varphi)$,

$$\oint_{\partial B(\epsilon,0)} \nabla\left(\frac{\alpha}{r}\right) \cdot d\mathbf{a} = 4\pi Gm \longrightarrow -4\pi\alpha = 4\pi Gm, \tag{1.25}$$

and $\alpha = -Gm$. Thus the potential for a point source at the origin is $\varphi = -Gm/r$, and the associated field is $\mathbf{g} = -\nabla\varphi = -Gm/r^2\hat{\mathbf{r}}$, a good check.

We can move the source to an arbitrary location $\mathbf{r}'$, and then the potential just has $r \to |\mathbf{r} - \mathbf{r}'| \equiv \mathbf{R}$. The solution to a partial differential equation, like Poisson's, for a point source described by a delta function is called the "Green's function solution" and is denoted (problematically for our present purposes) $G(\mathbf{r}, \mathbf{r}')$ for a source at $\mathbf{r}'$. We have just found the Green's function for the gravitational Poisson problem,[7]

$$G(\mathbf{r}, \mathbf{r}') = -\frac{G}{R}, \qquad \mathbf{R} \equiv \mathbf{r} - \mathbf{r}'. \tag{1.26}$$

The Green's function is useful in theories that support superposition, linear PDEs, since we can build solutions for arbitrary source distributions. Referring to Figure 1.3, the potential at $\mathbf{r}$ due to the sum of point contributions from $\rho(\mathbf{r}')$ is

$$\varphi(\mathbf{r}) = \int G(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}')\, d\tau', \tag{1.27}$$

where again the integral is over all space, and ρ is taken to be a function that is zero outside of the actual mass distribution. This integral solution is built in much the same way as (1.5), and rather than review the construction, we will content ourselves to check that $\varphi(\mathbf{r})$ satisfies (1.23). Hitting both sides of (1.27) with the Laplacian and noting that $\nabla^2 G(\mathbf{r}, \mathbf{r}') = 4\pi G\delta^3(\mathbf{r} - \mathbf{r}')$ by definition,

$$\begin{aligned} \nabla^2\varphi(\mathbf{r}) &= \int \left[\nabla^2 G(\mathbf{r}, \mathbf{r}')\right]\rho(\mathbf{r}')\, d\tau' \\ &= \int \left[4\pi G\delta^3(\mathbf{r} - \mathbf{r}')\right]\rho(\mathbf{r}')\, d\tau' = 4\pi G\rho(\mathbf{r}). \end{aligned} \tag{1.28}$$

The computational advantage in working with $\varphi(\mathbf{r})$ instead of $\mathbf{g}(\mathbf{r})$ is simplicity: a single function rather than the field, which is described by three functions. But it can also be used to get at different types of quantities than the field. As an example, let's compute the energy required to build a discrete distribution of mass, like the one shown in Figure 1.4. There are N masses $\{m_i\}_{i=1}^N$ positioned at $\{\mathbf{r}_i\}_{i=1}^N$. To find the work required to build the distribution, we'll

[6] This is conventional, and comes more from the notion of a gauge choice than any physical requirement. The potential energy $m\varphi$ has an undetectable offset that can be chosen however we like (only energy differences matter). If the field itself were nonzero at spatial infinity, we would have an actual observational problem, since there are no forces with infinite range.

[7] The Green's function traditionally omits the point mass m, so that $G(\mathbf{r}, \mathbf{r}')$ satisfies $\nabla^2 G(\mathbf{r}, \mathbf{r}') = 4\pi G\delta^3(\mathbf{r} - \mathbf{r}')$. I have highlighted the argument of the Green's function to avoid confusion with the gravitational constant G, which appears on the right in (1.26).