EQUIVALENTS OF THE RIEMANN HYPOTHESIS

Volume Three: Further Steps towards Resolving the Riemann Hypothesis

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Equivalents of the Riemann Hypothesis

Volume Three: Further Steps towards Resolving the Riemann Hypothesis

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University of Waikato, New Zealand

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Dedicated to Jackie, Jude and Beck

RH is a precise statement, and in one sense what it means is clear, but what it is connected with, what it implies, where it comes from, can be very unobvious.

Martin Huxley

Contents

Preface *page* xiii

1 Nicolas' $\pi(\mathbf{x}) < \mathrm{li}(\theta(\mathbf{x}))$ Equivalence 1
1.1 Introduction 1
1.2 Estimating the Logarithmic Integral 4
1.3 The Function $A_1(x)$ 11
1.4 The Functions $B(x)$ and $A_2(x)$ 16
1.5 Asymptotic and Explicit Bounds for the Function $A(x)$ 22
1.6 A Big Omega Theorem of Robin 28
1.7 End Note 34

2 Nicolas' Number of Divisors Function Equivalence 35
2.1 Introduction 35
2.2 Results Derived from Chapter 1 39
2.3 Superior Highly Composite Numbers 41
2.4 Preliminary Lemmas 49
2.5 Technical Lemmas 59
2.6 Proof of Nicolas' Explicit Inequality Assuming RH 72
2.7 Benefit and Convexity 74
2.8 The Main Theorem 76
2.9 End Note 87

3 An Aspect of the Zeta Function Zero Gap Estimates 88
3.1 Introduction 88
3.2 Preliminary Results 90
3.3 Gonek's Sum of Powers to Zeta Zeros 95
3.4 Main Theorem 101

4 The Rogers–Tao Equivalence 109
4.1 Introduction 109
4.2 Definitions and Overview 110
4.3 Preliminary Results 113
4.4 Counting the Zeros of $H_t(z)$ 133

4.5 A Lower Bound on Gaps Between Zeros 139
4.6 Asymptotics for the Integral of the Energy 143
4.7 Evolution of the Adjusted Hamiltonian 156
4.8 Estimates for the Hamiltonian and Energy 164
4.9 The Fundamental Lemma and Main Theorem 172

5 The Dirichlet Series of Dobner 175
5.1 Introduction 175
5.2 Preliminary Lemmas 178
5.3 Fundamental Lemma and Theorem 186
5.4 Main Result 196
5.5 Dobner's Theorem for an Extended Selberg Class 198
5.6 End Note 207

6 An Upper Bound for the de Bruijn–Newman Constant 208
6.1 Introduction 208
6.2 Imported Results 212
6.3 Definitions 215
6.4 Notation 218
6.5 Zero Dynamics 219
6.6 Basic Estimates 222
6.7 Fundamental Lemma and Theorem 225
6.8 Essential Estimates 240
6.9 The Unbounded Region 260
6.10 The Barrier Region 264
6.11 The Bounded Region 271
6.12 Criteria for an Upper Bound 279
6.13 The Main Theorem 284
6.14 End Note 288

7 The Pólya–Jensen Equivalence 289
7.1 Introduction 289
7.2 Jensen Polynomials and the Class $\mathscr{LP}$ 290
7.3 The Pólya–Jensen Equivalence for RH 293
7.4 The Work of Csordas et al. 298
7.5 The Result of Chasse 298

8 Ono et al. and Jensen Polynomials 307
8.1 Introduction 307
8.2 Hermite Polynomials 310
8.3 The Linear Growth Condition 313
8.4 Asymptotics for the $\xi(s)$ MacLaurin Coefficients 316
8.5 The Linear Growth Condition for the Coefficients 320
8.6 The Second Article of Ono et al. 323
8.7 The Results's Reception 324
8.8 David Farmer's Response 325

8.9 Conrey and Gosh's Example 332
8.10 End Note 334

9 Gonek–Bagchi Universality and Bagchi's Equivalence 337
9.1 Introduction 337
9.2 Notations 340
9.3 Gonek's Proof Overview 341
9.4 Gonek's First, Fundamental Lemma 343
9.5 Technical Lemmas 355
9.6 Gonek's Second Fundamental Lemma 365
9.7 Gonek's Universal Property Theorem 373
9.8 Corollaries to Gonek's Theorem 376
9.9 Bagchi's Lemma and RH Equivalence Overview 379
9.10 Bagchi's RH Equivalence 380
9.11 End Note 386

10 A Selection of Undecidable Propositions 388
10.1 Introduction 388
10.2 Poonen's List and Other Undecidable Examples 390
10.3 Semi-Thue Systems 394
10.4 Tag Systems 396
10.5 Hilbert's 10th Problem is Undecidable 397
10.6 Some Undecidable Consequences of DPRM 409
10.7 Laczkovich's Undecidable Example 412
10.8 Congruential Functions 416
10.9 Conway's Unpredictable Iterations 418

11 Equivalences and Decidability for Riemann's Zeta 424
11.1 Introduction 424
11.2 RH and the Arithmetic Hierarchy 426
11.3 Matiyasevich's Polynomial RH Equivalence 428
11.4 Matiyasevich's Integer Equivalence 435
11.5 Ramsey Theory 440
11.6 The Paris–Harrington and Sine Principles 444
11.7 Paris–Harrington Theorem Proof 446
11.8 Bovykin–Weiermann Preliminary Results 455
11.9 The Bovykin–Weiermann Theorems 461
11.10 Applications of Recursive Function Theory to RH 463
11.11 Epilogue 469

***Appendix A* Imports for Gonek's Theorems 471**

***Appendix B* Imports for Nicolas' Theorems 490**
B.1 Theorems 490
B.2 Superior Highly Composite Numbers 494

Appendix C **Hyperbolic Polynomials** 495
C.1 Introduction 495
C.2 Results of Schur, Marlo, and Obrechkoff 497
C.3 Borcea–Branden's and Chasse's Theorems 514
Appendix D **Absolute Continuity** 516
Appendix E **Montel's and Hurwitz's Theorems** 526
Appendix F **Markov's and Gronwall's Inequalities** 530
Appendix G **Characterizing Riemann's Zeta Function** 532
Appendix H **Bohr's Theorem** 541
Appendix I **Zeta and L-Functions** 544
I.1 Introduction 544
I.2 The Selberg Class 545
I.3 Properties of the Selberg Class 546
I.4 Selberg's Conjectures 547
I.5 Consequences of the Selberg Definitions and Conjectures 548
I.6 Selberg Data 549
I.7 Dokchitser L-functions 549
I.8 Iwaniec–Kowalski Class 550
I.9 Consequences of These Conditions 551
I.10 Examples 552
Appendix J **de Reyna's Expansion for the Hardy Contour** 553
J.1 Riemann–Siegel Formula 553
J.2 Riemann's Integral for Zeta 554
J.3 Arias De Reyna's Expansion 560
Appendix K **Stirling's Approximation for the Gamma Function** 565
K.1 Introduction 565
K.2 Polymath15's Estimate 565
Appendix L **Propositional Calculus $\mathscr{P}_0$** 567
L.1 Introduction 567
L.2 A Brief Account of the Beginnings of Mathematical Logic 567
L.3 Propositional Calculus 570
L.4 The System $\mathscr{P}_0$ 573
Appendix M **First Order Predicate Calculus $\mathscr{P}_1$** 579
M.1 Introduction 579
M.2 First, Order Mathematical Theories 580
M.3 Examples of First-Order Theories 583
M.4 Models and Truth in an Interpretation 585
M.5 Logical Axioms and Rules of Inference for $\mathscr{P}_1$ 587
M.6 Theorems for $\mathscr{P}_1$ 591

M.7 Decidability in $\mathscr{P}_1$ 596
M.8 Some Mathematical Applications 602
M.9 Models and the Compactness Theorem 606
M.10 Gödel’s Incompleteness Theorem 608
M.11 Arithmetic 614
M.11.1 Peano Arithmetic PA 614
M.11.2 Compact Arithmetic CA 616
M.11.3 Presburger Arithmetic PR 616
M.11.4 Skolem Arithmetic SK 617
M.11.5 Robinson Arithmetic Q 618
M.11.6 Takeuti’s Conservative Extension PAT 618
M.12 Arithmetical Hierarchy 624

Appendix N **Recursive Functions** 626
N.1 Introduction 626
N.2 Partial Recursive and Primitive Recursive Functions 627
N.3 Decidable Predicates 630
N.4 Recursively Enumerable Subsets 631
N.5 Enumeration and Rice’s Theorem 633
N.6 Algorithms and Machines 636
N.7 Turing Machines 637
N.8 Minsky Machines 641

Appendix O **Ordinal Numbers and Analysis** 649
O.1 Introduction 649
O.2 Ordinal Numbers 649
O.3 Primitive Recursive Arithmetic PRA 652
O.4 Gentzen’s Consistency of Arithmetic 654
O.5 The Ordinal Strength of Theories 656
O.6 Paris–Harrington Theorem Proof using Ordinals 656
O.7 End Note 664

References 665

Index 677

Preface

Recent developments have provided strong motivation for writing a third and last volume in this series of works. It includes some recent equivalents to the Riemann hypothesis (RH) published in the main during the second decade of this century. Two are arithmetical, one quite analytic, and one very computational. Given the Riemann zeta function is defined using at least Peano arithmetic, and that form of arithmetic is undecidable, there must be a range of undecidabilities waiting to be found within its family of properties. To this end I have concluded the volume with what I hope is a sufficient set of examples and structures of mathematical logic to stimulate research. After more than 160 years of trying without success, this approach could offer some insights at least into why resolving the Riemann hypothesis is such a difficult problem. Even though, as revealed in the penultimate chapter, there are a wide range of undecidable problems in mathematics, the evidence presented in the final chapter is that, as expected, RH is in some sense decidable.

In Volume One there are equivalences for the arithmetical functions $\psi(x)$, $\theta(x)$, $\sigma(n)$, $\pi(x)$ and $\varphi(n)$, but none for the number of divisors function $d(n)$. This omission has now been rectified through the work of Jean-Louis Nicolas. His result on the function $d(n)$ is dependent on another of his equivalences, the inequality $\pi(x) < \mathrm{li}(\theta(x))$. An account of these two closely related equivalences is derived in Chapters 1 and 2. The main chapter and appendix dependencies for Nicolas' results are given in Figure 2.

In Chapter 5 of Volume Two there is an account of work on the de Bruijn–Newman constant Λ, where RH is equivalent to $\Lambda \leq 0$. Csordas et al. had shown there is an explicitly computable $\epsilon > 0$ such that $-\epsilon < \Lambda$, with considerable effort being spent over an extended period of time to reduce the size of ϵ. It came as a nice surprise in 2018 when Brad Rogers and Terence Tao, using methods based on those of Csordas et al., proved that $\Lambda \geq 0$. Thus, RH is equivalent to $\Lambda = 0$. Their proof contains many new ideas and is set out in Chapter 4.

The proof, for its ultimate step, requires an upper estimate, in terms of the height, for a positive proportion of the zeta zero height gaps due to Brian Conrey, Amit Gosh, Steve Gonek, Roger Heath-Brown and others. This is derived in Chapter 3. The main dependencies for the result of Rogers and Tao are given in Figure 1.

Soon after the Rogers/Tao work was published on the archive, a much shorter method for obtaining the same result $\Lambda \geq 0$ was found by Alexander Dobner. His method applies not only to the Riemann zeta function but also to the extended Selberg class of L-functions. That this extension is valuable can be seen in the final sections of Chapter 8. "Dobner's contour", Figure 5.3, lies at the heart of his method. This is described for $\zeta(s)$ and outlined for the extended Selberg class in Chapter 5. The Selberg class is described in Appendix I, with the main difference between the class and extended class being the absence in the latter of an Euler product requirement. Siegel's nice proof of Hamburger's theorem, which shows the product is not really needed to characterize the Riemann zeta function is given in Appendix G. It underpins the importance of the extended class.

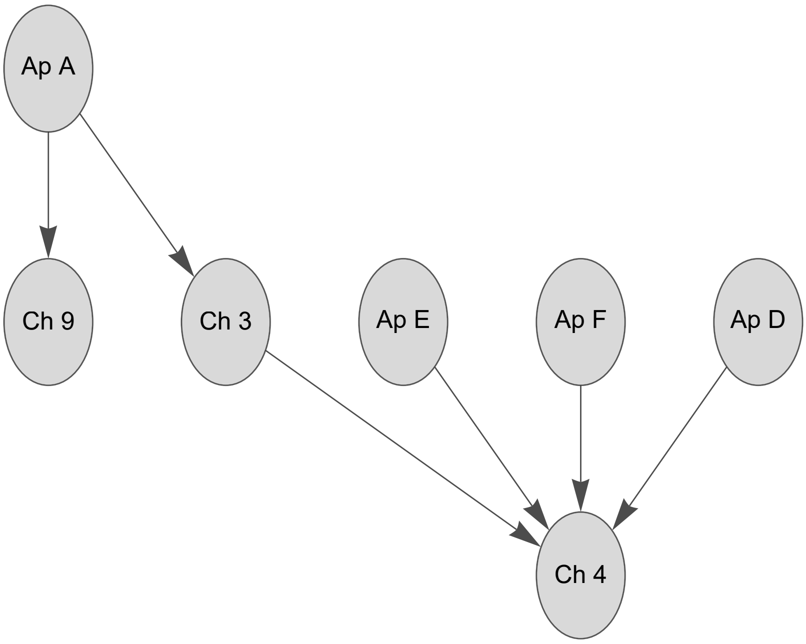

Figure 1 Dependencies for Chapter 4.

Chapter 7: buried somewhat in two papers from 1913 written in German, one by Jensen and the other by Pólya, is an equivalence of these two authors that might very well have been included in Volume Two. This equivalence uses the Laguerre–Pólya class of [30, chapter 5], and will be derived in Chapter 7. It relates to the zeros of polynomials with real coefficients formed by truncating a Taylor expansion of the Riemann $\Xi(s)$ function being all real. This was shown to be true in 2011 by Chasse for all Jensen polynomial degrees up to 2×10^{17}, and his proof is included in Chapter 7. It relies on Platt's value for critical zeta zero heights, and properties of polynomials given by Obrechkoff, Schur, Walsh, Borcea, Brändén and others. These properties are set out in Appendix C.

Following the success of Rogers and Tao in obtaining the equivalence $\Lambda \geq 0$, a Polymath group, Polymath15, was established by Terence Tao with the goal of improving the known upper bound $\Lambda < 0.5$. The results of this work are given in Chapter 6. In a very short period of time, experimenting with more than half a dozen computer languages to obtain acceptable efficiency, the group arrived at an upper bound $\Lambda \leq 0.22$. Their work was based on the earlier work on Λ, especially that of de Bruijn set out in [30, section 5.3]. Note that in the notation of Volume Two [30, table 5.2], Polymath15 uses Λ_C for Λ.

Not having the high-powered computational expertise of some of the members of the Polymath group, this author struggled to reproduce their results, but managed to do so using the language Julia. This is in contrast to Polymath who used Pari and C for their final calculations. The Polymath work, as does that of Chasse on Jensen polynomials in Section 7.5, depends on a currently well regarded height due to David Platt, up to which all zeta zeros in the critical strip lie on the critical line. It may well be that further improvements in the Λ upper bound will depend on increases in this height.

However, it just might be possible to use the analytic part of Polymath15's techniques, as they stand, to show that one could shrink the critical strip by some explicit $\epsilon > 0$, or even to demonstrate that such an ϵ must exist. That would be great progress!

The language Julia has many excellent modern features, and could in the future reach a level comparable to LaTeX in terms of value and robustness. It is open source, fast, has a just-in-time compilation step, can employ a wide range of symbols, and has many attached packages. Currently it tends to change rather often – it is quite a new system, so has not been included in the form of a package for Volume Three.

Chapter 8: in addition RH is equivalent to the stronger statement that some associated polynomials, namely all so-called Jensen polynomials related to the Riemann $\Xi(s)$ function, have all real roots for all non-negative shifts of the coefficients. Again it came as a delightful surprise to learn, in 2018 when the second author was in NZ, of the new result of Griffen, Ono, Rolen and Zagier that each Jensen polynomial has all real roots for all shifts sufficiently large

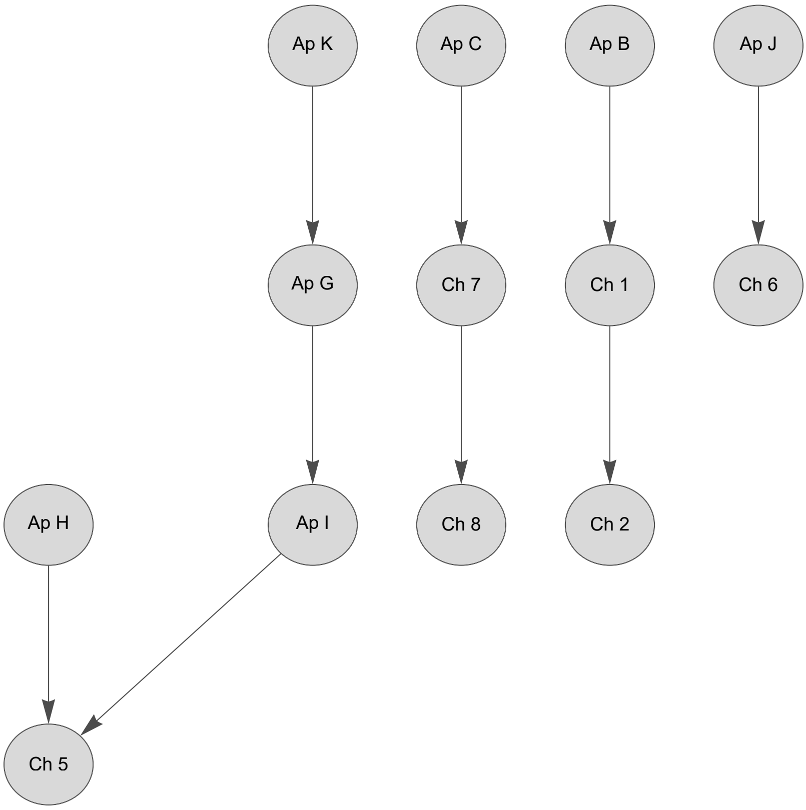

Figure 2 Dependencies for Chapters 2, 5, 6 and 8.

depending on the degree $n \geq N(d)$, with $N(d)$ explicit but not optimal. This was shown in a second paper by a related group of authors. The result depends on estimates and properties of Hermite polynomials, which are derived in the chapter.

However, a survey paper by David Farmer, outlined in Section 8.8, gives a critique of the Griffen et al. work, sets it in a broad context of earlier work, makes a case that their results are more analysis than number theory, and shows why their approach won't be particularly efficient when it comes to checking zeros. Indeed, he gives a heuristic which tends to show that Jensen polynomials are significantly less efficient in finding zeros off the critical line than Taylor polynomials. We neither affirm or attempt to negate this critique, but believe it provides valuable insights into not only the Griffen et al. work, but also new work relating to the de Bruijn–Newman constant reported in Chapter 6, especially the work of Dobner. In this regard the example of Conrey and Gosh based on the extended Lindelöf hypothesis and the extended

Selberg class is set out at the end of the chapter. It illustrates the usefulness of going beyond zeta.

Given the difficulties experienced by many mathematicians in attempting to resolve RH over the past 160 years, the question naturally arises whether it might be "undecidable". Readers will be familiar with some undecidable questions or theories, but it seemed helpful to include a chapter showing the broad scope of these examples. This is Chapter 10, which has been built starting with a list of Bjorn Poonen, but includes the Martin Davis proof of the "unsolvability" of Hilbert's 10th problem, undecidabilities which can be derived from this result, and other proofs, to form a basis of ideas on what sort of mathematical questions might be undecidable.

Some of the applications of the negative solution to Hilbert's 10th problem use topological density methods. As far as the Riemann zeta function is concerned, these are analogous to the universality theorems of Voronin. Here I include an exposition of a very general form of universality for $\zeta(s)$ based on the thesis of Steve Gonek, and independently Bhaskar Bagchi. This is in the setting of joint universality for finite families of Dirichlet L-functions, and is given in Chapter 9. That chapter also includes an application to a very nice equivalence of Bagchi which is based on a dynamical systems concept, strong recurrence, although the reader does not require a background in that subject. Bagchi's valuable positive upper density result for translates in zeta's universal property is also given.

Included in this volume are expository information and tools of mathematical logic, set out in four appendices. These are provided to assist readers in exploring the "is RH undecidable or decidable?" issue. Consulting a sample of many of the existing texts on introductory mathematical logic, covering the classical results in the field from mostly the twentieth century, it was hard to pick and choose definitions and results from the sample, since not all authors agreed. For example fundamental to this quest is the definition of "recursive function" so this at least needed to be solid and effective. I chose the texts I found most satisfactory, and included a summary, with some proofs and examples, of some parts of Richard Hodel's book *An Introduction to mathematical logic* [112], as part of the appendices L on the propositional calculus and M on the predicate calculus. For the appendix on recursive functions N, the material has been based in part on Nigel Cutland's text *Computability* [66]. The final appendix O gives the reader a taste of proof theory and the opportunity to consider the goal of analyzing and comparing proofs of RH equivalences using some of the tools developed during the twentieth century.

The main chapter in the part of the book which deals with equivalences based on concepts from mathematical logic is Chapter 11. The dependencies are given in Figure 3.

We begin in Section 11.2 with the so-called arithmetic hierarchy. This is defined later in Appendix M.12. A variety of RH equivalent statements, which take the form of first-order predicate calculus statements at the base of the hierarchy, are summarized.

Two equivalences of Matiyasevich follow in Sections 11.3 and 11.4. The first shows how a single polynomial in many variables and integer coefficients may be derived, using the equivalence of Shapiro [29, section 10.1], such that RH is equivalent to the polynomial having no integer solution. The second is an RH equivalence which takes the form of an inequality involving arithmetic functions which have only integer values. Then RH is equivalent to the inequality holding for all $n \in \mathbb{N}$. A related computer algorithm halts if and only if RH is false, showing RH is at least semi-decidable.

Next, we embark on a description of some ideas which have been very influential in the derivation of arithmetic/combinatorial statements which are true but cannot be proved in PA, a restricted form of the Peano axioms for arithmetic. This includes the graph theory of König, the examples of Paris and Harrington, and the applications to $\zeta(s)$ of Bovykin and Weiermann set out in part in Sections 11.5, 11.6, 11.8 and 11.9. We include the proof of the Dirichlet–Kronecker density theorem which plays an essential role. The theorem of Paris and Harrington is based on a simple combinatorial statement which is true but cannot be proven in PA. This is used in the theorem of Bovykin and Weiermann to show that for a given $\sigma > 1$ the statement that there are particular n-tuples of integers with products a, b such that the zeta values $\zeta(\sigma + ia)$ and $\zeta(\sigma + ib)$ are close in a particular manner, is true but cannot be proved in PA. There is also a form of the statement for $\frac{1}{2} < \sigma \leq 1$. This shows there are explicit undecidabilities in $\zeta(s)$.

In the final section of Chapter 11 we set out a number of possibly plausible approaches to RH using techniques from mathematical logic. This is set out by way of eight examples, which are really exercises for the reader to refute or elaborate to improve the level of rigour:

(1) An undecidable set of zeta-like functions which includes $\zeta(s)$ defined using Euler products and partial recursive functions, assuming RH is false.

(2) Again assuming RH is false, an undecidable set of partial recursive functions giving the multiplicities of zeta zeros.

(3) A demonstration which shows that if RH is true the zeros without multiplicity are recursively enumerable, as are the zeros with multiplicity.

(4) An analytic model of an arbitrary Collatz function, and thus any Minsky or Turing machine, using one of zeta's universal properties.

(5) Similarly, a model for Conway's rational or vector games using $\zeta(s)$.

(6) Potential use of the method of Richardson/Caviness/Wang to find undecidabilities in zeta. Application of the method of Bovykin and Weiermann.

(7) Again assuming RH is true and that all zeros are simple, we can decide

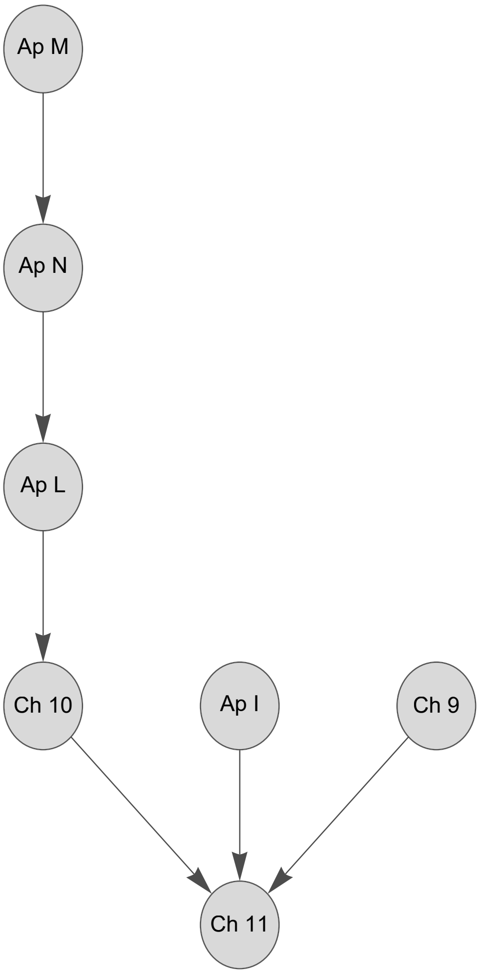

Figure 3 Some dependencies for Chapter 11.

whether γ is not the imaginary part of a zeta zero using the holomorphic flow of $\xi(s)$.

(8) Using Theorem N.10, which states a set $A \subset \mathbb{N}_0$ is decidable if and only if it is the image of a partial recursive function, which is defined everywhere and strictly increasing, we show that if we assume all zeros on the critical line are simple, then RH is decidable.

I gratefully acknowledge the help and support given to me by the University of Waikato, including Daniel Delbourgo, Tim Stokes, and the forever patient members of the computer support groups. Assistance from the mathematics community was given by Alex Dobner, Steve Gonek, Michael Griffin, Yuri Matiyasevich, Jean-Louis Nicolas, Ken Ono, Brad Rogers, Dinesh Thakur,

and Andreas Weiermann. I am also indebted to Richard Hodel and Nigel Cutland and for their beautiful expositions of aspects of mathematical logic which I drew on for material in several appendices. Finally I must acknowledge the very significant use that has been made of the work of Gaisi Takeuti on conservative extensions of Peano arithmetic PA, bridging the gap between the complex functions used in number theory and mathematical logic.

Without the computer systems, Julia, Mathematica, and TikZ, much of the work could not have progressed and been explained as it has been. These are both fine and extensive computer software entities representing the best of the old and new worlds for mathematical computing. Throughout this writing project Cambridge University Press, especially Roger Astley and Clare Dennison, the three copy editors and the Cambridge technical support people, have been enormously encouraging and helpful. Lastly, my solace and support Jackie has never failed to believe I could complete this task with its multiple challenges, even as we both moved into elderhood.