1

Nicolas' $\pi(\mathbf{x}) < \mathrm{li}(\theta(\mathbf{x}))$ Equivalence

1.1 Introduction

To begin this introduction, we give a summary of results for two inequalities which are closely related to the inequality of Jean-Louis Nicolas, which is the subject of this chapter. Numerical evaluation up to modest values of x gives $\pi(x) < \mathrm{li}(x)$. It was thought by many in the early part of the twentieth century that this might always be the case. Given the prime number theorem (PNT) estimate

$$\pi(x) = \mathrm{li}(x) + O\left(x \exp\left(-c\sqrt{\log x}\right)\right),$$

Nicolas' inequality would have provided a useful simplification. However, in 1914 Littlewood showed, using a method developed by Landau, that $\mathrm{li}(x) - \pi(x)$ changed sign infinitely often as $x \to \infty$ [116, chapter V]. Littlewood's research student Skewes set about finding the first number for which $\mathrm{li}(x) < \pi(x)$. In 1933, assuming RH, Skewes showed that such a number would not be greater than

$$10^{10^{10^{34}}}.$$

He continued to work on this problem and by 1955 had shown, unconditionally, that the number would need to be no greater than the astronomical

$$10^{10^{10^{964}}}.$$

Many number theorists were fascinated by this problem and progressively reduced the proved upper bound, or found an interval in which there was at least one zero crossing for $\mathrm{li}(x) - \pi(x)$. They included Lehman (1966), te Riele (1987), Bays and Hudson (2000), Chao and Plymen (2010), Saouter and Demichel (2014), Zegowitz (2010), and Stoll (2011).

For the initial interval of positivity, J. B. Rosser and L. Schoenfeld (1962) [206] showed that $\pi(x) < \mathrm{li}(x)$ continued to hold at least up until 10^8. R. Brent (1975) [24] improved this to 8×10^{10}, T. Kotnic (2008) [129] to 10^{14}, D. J. Platt and T. S. Trudgian (2016) [188] to 1.39×10^{17}, and J. Büthe (2017) [39] to 10^{19}. We note Littlewood's theorem of 1914 reveals there is an infinite number of crossings [116, theorem 35]. It takes the form

$$\mathrm{li}(x) - \pi(x) = \Omega_{\pm}\left(\frac{\sqrt{x}\,\mathrm{logloglog}\,x}{\log x}\right).$$

Michael Rubinstein and Peter Sarnak in 1994 [208] showed that the logarithmic density of positive integers for which $\mathrm{li}(x) < \pi(x)$ exists and is about 2.6×10^{-7} of all integers.

The difference $x - \theta(x)$ has a similar set of behaviours, although not as extensively studied as $\mathrm{li}(x) - \pi(x)$. The method of Landau, when applied to $x - \psi(x)$, because

$$\psi(x) = \theta(x) + \theta(x^{1/2}) + O(x^{1/3+\epsilon}),$$

can be used to show $x - \theta(x)$ changes sign infinitely often as $x \to \infty$. Indeed, more precisely [116, theorem 33]

$$x - \theta(x) = \Omega_{\pm}\left(x^{1/2-\epsilon}\right).$$

Regarding the initial interval, Schoenfeld (1976) showed that $\theta(x) < x$ up to 10^{11}, Dusart (2010) to 8×10^{11}, and Platt and Trudgian in Theorem B.2 (2015) that there is an

$$x \in [e^{x_0-h}, e^{x_0+h}],\ x_0 = 727.951332655,\ h = 1.3 \times 10^{-8},$$

for which $x < \theta(x)$.

It came as a surprise to the author that the "irregularities of distribution" ([116, chapter V]) exhibited by the three functions $\pi(x)$, $\mathrm{li}(x)$ and $\theta(x)$ would give rise to an RH equivalence. Indeed, that the functions might conspire together to give an inequality closely related to $\theta(x) < x$ and $\pi(x) < \mathrm{li}(x)$, which was true on an unbounded interval if RH was true, but alternated between true and false infinitely if RH was false. This result was published by Jean-Louis Nicolas in 2017 [172] and has the statement

$$RH \iff \pi(x) < \mathrm{li}(\theta(x)),\ x \geq 11.$$

The proof is set out in this chapter as Theorem 1.17. Consistent with $\pi(x) < \mathrm{li}(x)$ and $\theta(x) < x$ the proof in the RH is false case, gives not just one counterexample but an infinite set x_n of counterexamples with $x_n \to \infty$. In the RH is true case $\mathrm{li}(\theta(x)) - \pi(x)$ is not only positive but has limit value infinity. This can be derived from a different equivalence of Nicolas, stated in an end note to the chapter.

To prove his result Nicolas defines the difference $A(x) = \mathrm{li}(\theta(x)) - \pi(x)$ and splits it into two parts using the function $\Pi(x)$. The definitions follow:

$$\begin{aligned}
\Pi(x) &:= \sum_{p^j \le x} \frac{1}{j} = \sum_{j=1}^{\left\lfloor \frac{\log x}{\log 2} \right\rfloor} \frac{\pi(x^{1/j})}{j},\\
A_1(x) &:= \operatorname{li}(\psi(x)) - \Pi(x),\\
A_2(x) &:= \operatorname{li}(\theta(x)) - \operatorname{li}(\psi(x)) + \Pi(x) - \pi(x),\\
A(x) &:= \operatorname{li}(\theta(x)) - \pi(x) = A_1(x) + A_2(x).
\end{aligned}$$

The intricate detailed relationships between the lemmas required to prove the theorem are described in Figure 1.1. Note the important role played by the imported results set out in Appendix B.

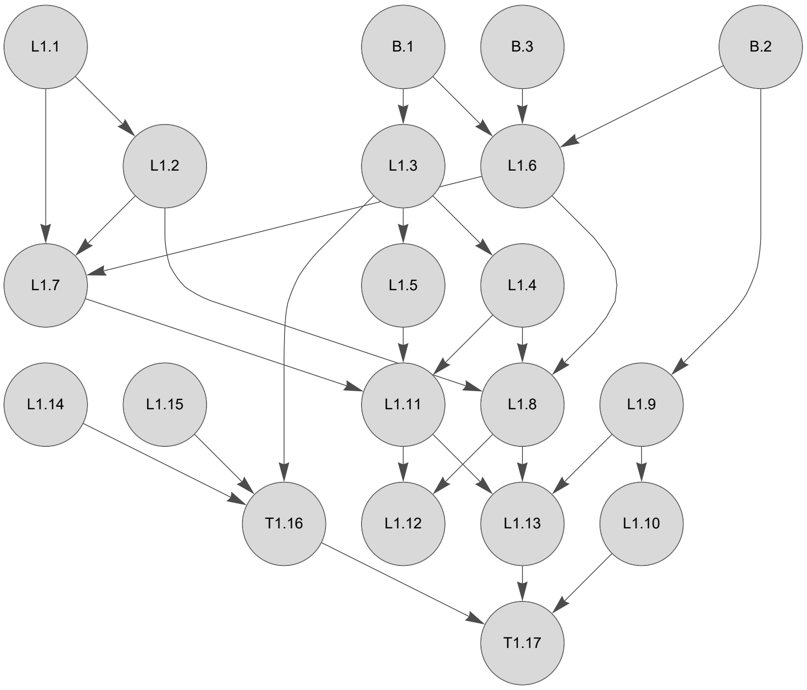

Figure 1.1 Dependencies for Theorem 1.17.

We don't develop the fascinating consequences of Nicolas' theorem, such as if we assume RH is true we get

$$\theta(x) < x \implies \pi(x) < \operatorname{li}(x).$$

Because of this, the first crossing point for x and $\theta(x)$, under RH, must come before that of $\pi(x)$ and $\operatorname{li}(x)$, and the reverse is true for the second one. Any density which exists for $\pi(x) - \operatorname{li}(x)$ must be no greater than that for $\theta(x) - x$.

In Section 1.2 we estimate $\operatorname{li}(x)$, in Section 1.3 the function $A_1(x)$, in Section 1.4 $A_2(x)$, and in Section 1.5 the function $A(x)$, all assuming RH is true. Where it is needed, we use the equivalence of Schoenfeld given in Volume One and quoted in this volume in Appendix B. Then for the case RH is false we first prove part of Guy Robin's result, Theorem 1.16 which is

$$A(x) = \Omega_-(x^\alpha),\ 0 < \alpha < \Theta,$$

where $\Theta := \sup\{\beta : \zeta(\beta + i\gamma) = 0\} > \frac{1}{2}$, which is all we need. This is then used to easily complete the proof of the equivalence, which is a little weaker than the result of Nicolas.

1.2 Estimating the Logarithmic Integral

First, we define the logarithmic integral valid for all $x > 1$ using the Cauchy principal value:

$$\mathrm{li}(x) := \lim_{\epsilon \to 0+} \int_0^{1-\epsilon} \frac{dt}{\log t} + \int_{1+\epsilon}^x \frac{dt}{\log t},$$

so

$$\mathrm{li}(x) := \mathrm{li}(2) + \int_2^x \frac{dt}{\log t},$$

with $\mathrm{li}(2) = 1.045163780117....$

For $x \to \infty$ we have the asymptotic expansions for the logarithmic integral valid for all $N \in \mathbb{N}$:

$$\begin{aligned}
\mathrm{li}(x) &= \sum_{j=1}^{N} \frac{(j-1)!x}{(\log x)^j} + O\left(\frac{x}{(\log x)^{N+1}}\right) \\
&= \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right) \\
&= \frac{x}{\log x} + \frac{x}{(\log x)^2} + O\left(\frac{x}{(\log x)^3}\right).
\end{aligned}$$

To see this note that by splitting the integral at $\sqrt{x}$ we get for $n \in \mathbb{N}$

$$\int_0^x \frac{1}{(\log x)^n}\, dx = O\left(\frac{x}{(\log x)^n}\right).$$

The expansion follows using integration by parts. In Figure 1.2 we show $\mathrm{li}(x)$ around its singularity, and in Figure 1.3 we give $\mathrm{li}(x)$ and its asymptotic approximation

$$\frac{x}{\log x} + \frac{x}{(\log x)^2} + \frac{2x}{(\log x)^3},$$

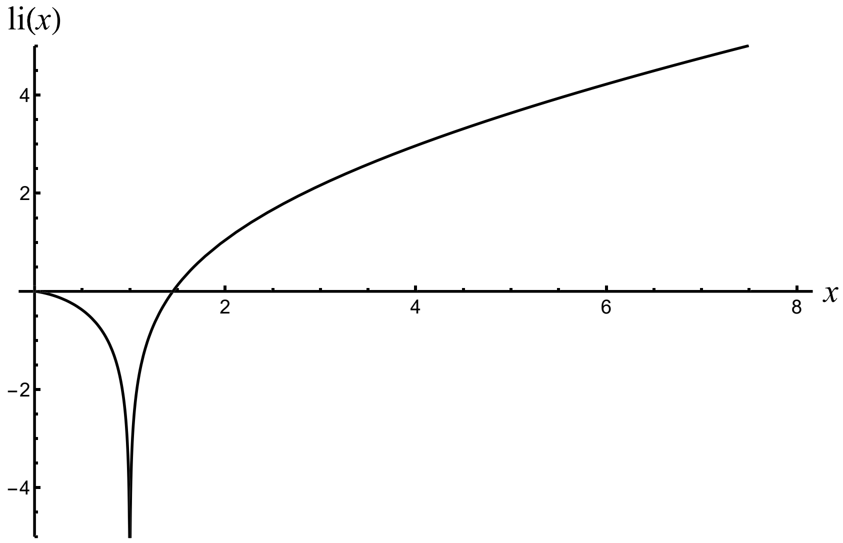

Figure 1.2 A plot of li(x) for $0 \leq x \leq 8$.

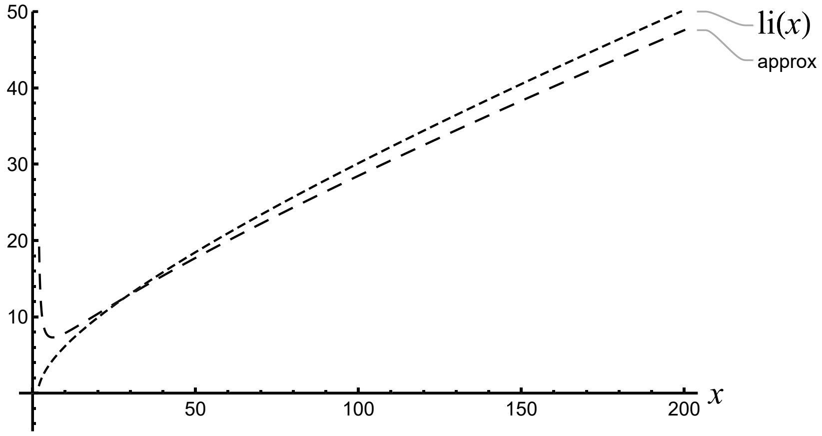

Figure 1.3 A plot of li(x) and an approximation for $2 \leq x \leq 200$.

which for at least $x \geq 20$ is less than li(x).

We note that the finite sum approximations are increasing with the number of terms and all terms, even the error for x sufficiently large, are positive for $x \geq 2$.

We use in the sequel the following functions relating to the difference between li(x) and its asymptotic expansions. We need only go to the second order:

$$L_1(x) := \mathrm{li}(x) - \frac{x}{\log x},$$

$$L_2(x) := \mathrm{li}(x) - \frac{x}{\log x} - \frac{x}{(\log x)^2},$$

$$F_1(x) := \frac{(\log x)^2\,\mathrm{li}(x) - x(\log x)}{x} = L_1(x)\frac{(\log x)^2}{x},$$

$$F_2(x) := \frac{(\log x)^3\,\mathrm{li}(x) - x(\log x)^2 - x(\log x)}{x} = L_2(x)\frac{(\log x)^3}{x}.$$

With these definitions we will see that $F_1(x)$ and $F_2(x)$ are bounded and have well-defined asymptotic limits.

Lemma 1.1 *The function $F_1(x)$ has the following and no other zeros or critical points on $[1,\infty)$:*
(i) $\lim_{x\to 1+} F_1(x) = 0$.
(ii) An absolute minimum at $x_3 = 1.85\ldots$ with value -0.488.
(iii) A positive zero at $x_0 = 3.8464\ldots$.
(iv) An absolute maximum at $x_4 = 94.6\ldots$ with value $1.784\ldots$.
(v) $\lim_{x\to\infty} F_1(x) = 1$.
In addition
(vi) For all $x > 1$ we have $\mathrm{li}(x) < 3x/4$.

Proof **(1)** First, note that for $x > 1$ we have the Taylor expansion

$$\mathrm{li}(x) = \log\log x + \gamma_0 + \sum_{n=1}^{\infty} \frac{(\log x)^n}{n \cdot n!}.$$

Since the sum is $O((x-1)e^{x-1})$, we can write as $x \to 1+$, $\mathrm{li}(x) = \log\log x + \gamma_0 + o(1)$. Thus, using l'Hôspital's rule to derive

$$\lim_{x\to 1+} (\log x)\log\log x = \lim_{y\to 0+} y\log y = \lim_{y\to 0+} \frac{\log y}{1/y} = -\lim_{y\to 0+} y = 0,$$

we get

$$\lim_{x\to 1+} F_1(x) = \frac{1}{x}\Big((\log x)^2(\log\log x + \gamma_0 + o(1)) - x\log x\Big)$$
$$= \lim_{x\to 1+} \frac{(\log x)}{x}(\log x)\log\log x = 0.$$

This proves (i).

(2) We now define three related functions which will enable the properties of $F_1(x)$ to be deduced:

$$f_1(x) := \frac{x^2}{\log x}F_1'(x),$$

$$
\begin{aligned}
&= 2\,\mathrm{li}(x) + x - \frac{x}{\log x} - \log(x)\,\mathrm{li}(x),\\
f_2(x) &:= x f_1'(x),\\
&= -\left(\mathrm{li}(x) - \frac{x}{\log x} - \frac{x}{(\log x)^2}\right) = -L_2(x) = -F_2(x)\frac{x}{(\log x)^3},\\
f_3(x) &:= f_2'(x) = -\frac{2}{(\log x)^3}.
\end{aligned}
$$

Figures 1.4 and 1.5 indicate how the first two functions behave.

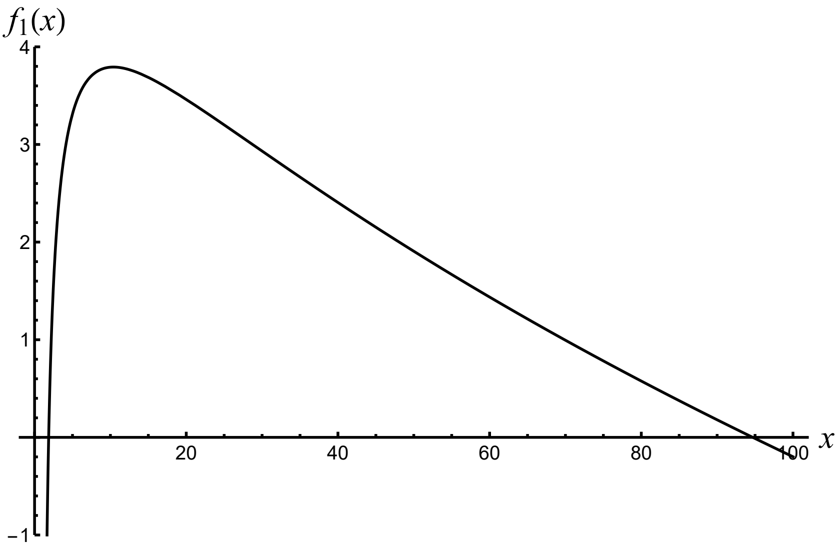

Figure 1.4 A plot of $f_1(x)$ for $2 \le x \le 100$.

Note that since $x > 1$, $f_2(x)$ and $f_1'(x)$ have the same sign, and that $f_3(x)$, hence $f_2'(x)$, is strictly negative. Thus, $f_2(x)$ is decreasing. Also the limit of $f_2(x)$ at $1+$ is $+\infty$ and at ∞ is $-\infty$. Therefore $f_2(x)$ has a unique zero in $(1,\infty)$ which we compute as $x_2 = 10.3973\ldots$. See Figure 1.5.

(3) We also derive

$$
\begin{aligned}
\lim_{x\to\infty} F_1(x) &= \lim_{x\to\infty} \frac{(\log x)^2\left(\frac{x}{(\log x)} + \frac{x}{(\log x)^2} + O\left(\frac{x}{(\log x)^3}\right)\right) - x(\log x)}{x}\\
&= \lim_{x\to\infty} \frac{x + O(x/(\log x))}{x} = 1.
\end{aligned}
$$

This proves (v).

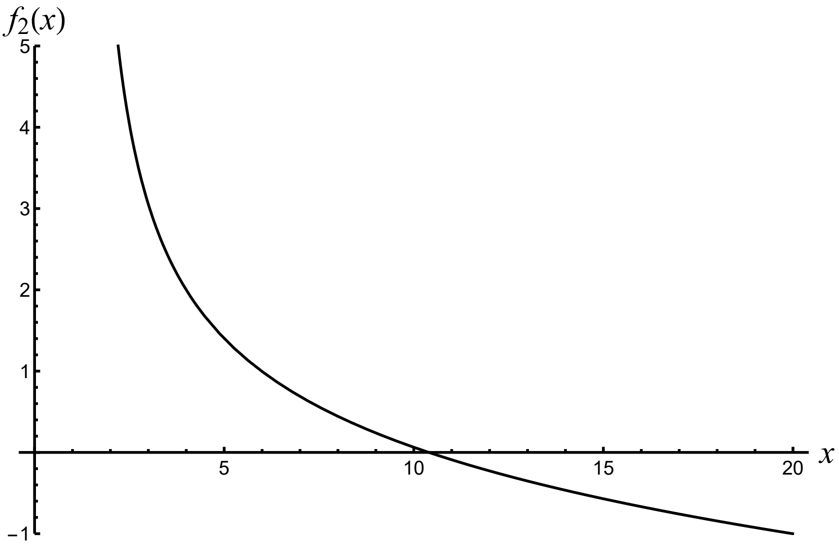

Figure 1.5 A plot of $f_2(x)$ for $1 \le x \le 20$.

(4) A computation shows $f_1(x)$ has precisely two zeros on $(1, \infty)$, at $x_3 = 1.85...$ and $x_4 = 94.6....$ Hence $F_1(x)$ has two corresponding critical points. Thus, we can say, moving from left to right, $F_1(1) = 0$, then $F_1(x)$ decreases to its minimum $F_1(x_3)$, then increases to its maximum $F_1(x_4)$, passing through a zero which we compute as $x_0 = 3.846467717...$, and then descends to its asymptotic limit 1 at ∞. Thus, we have (ii) and (iv). See Figures 1.6 and 1.7.

(5) Because

$$\frac{d}{dx}\left(\frac{\mathrm{li}(x)}{x}\right) = -\frac{F_1(x)}{x(\log x)^2}$$

is positive for $1 < x < x_0$ and negative for $x_0 < x$, $\mathrm{li}(x)/x$ has a maximum at x_0, and so we can write for all $x > 1$

$$\frac{\mathrm{li}(x)}{x} \le \frac{\mathrm{li}(x_0)}{x_0} \le 0.743 < \frac{3}{4},$$

so $\mathrm{li}(x) < 3x/4$. This proves (vi).

(6) In addition note that in the range $x > x_3$ we have $F_1(x) > 1$ so

$$\mathrm{li}(x) - \frac{x}{\log x} = L_1(x) = F_1(x)\frac{x}{(\log x)^2} > \frac{x}{(\log x)^2}$$

and so

$$\mathrm{li}(x) > \frac{x}{\log x} + \frac{x}{(\log x)^2}, \, x > x_3.$$

□

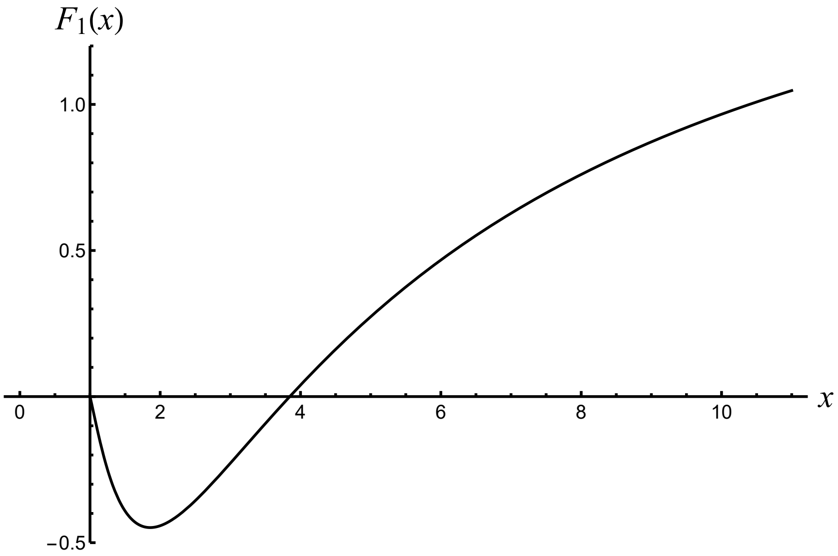

Figure 1.6 A plot of $F_1(x)$ for $1 \leq x \leq 11$.

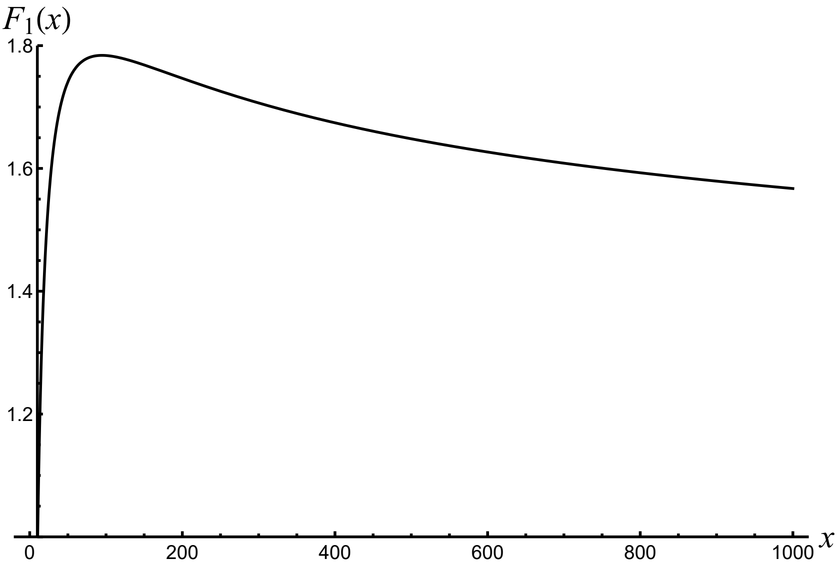

Figure 1.7 A plot of $F_1(x)$ for $11 \leq x \leq 1000$.

The function $F_2(x)$ behaves, qualitatively, in the same manner as $F_1(x)$. This gives rise to the possible use of higher-order approximations, $F_n(x)$, if needed.

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Excerpt
More Information

Lemma 1.2 *The function $F_2(x)$ has the following and no other zeros or critical points:*

(i) $\lim_{x\to 1+} F_2(x) = 0$.
(ii) An absolute minimum at $x_3 = 3.38\ldots$ *with value* $-1.369496\ldots$.
(iii) A positive zero at $x_0 = 10.39\ldots$.
(iv) An absolute maximum at $x_4 = 380.15\ldots$ *with value* $4.040415\ldots$.
(v) $\lim_{x\to\infty} F_2(x) = 2$.

Proof The proof is similar to that of Lemma 1.1. In this case we define

$$\begin{aligned} f_1(x) &:= \frac{x^2 F_2'(x)}{(\log x)^2}, \\ f_2(x) &:= x f_1'(x), \\ f_3(x) &:= f_2'(x) = -\frac{6}{(\log x)^4} < 0, \end{aligned}$$

and proceed using the same steps as in that lemma. The function $F_2(x)$ is plotted in Figures 1.8 and 1.9. □

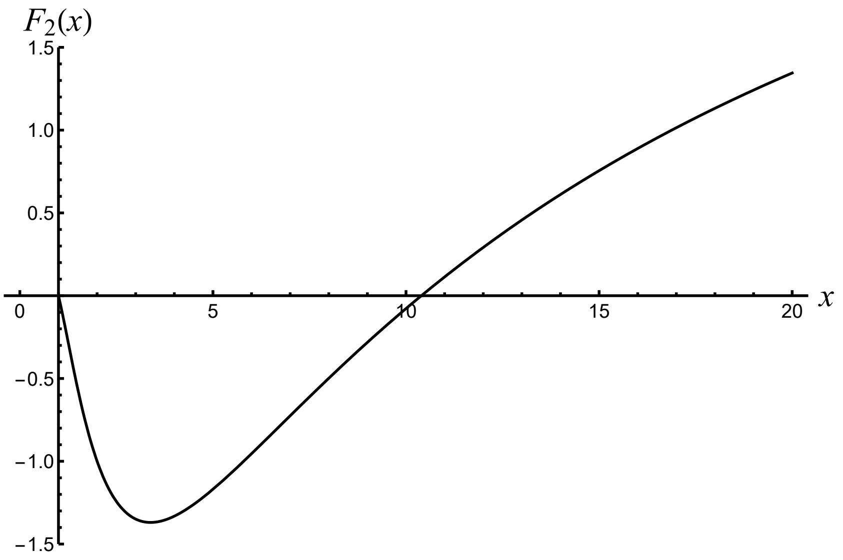

Figure 1.8 A plot of $F_2(x)$ for $1 \le x \le 20$.