Cambridge University Press & Assessment
978-1-009-38147-5 — A First Course in Magnetohydrodynamics
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Introduction

At last, some remarks are made about the transfer of momentum from the sun to the planets, which is fundamental to the theory. The importance of magnetohydrodynamic waves in this respect are [sic] *pointed out.*

First published mention of the term *magnetohydrodynamic*, from "On the cosmogony of the solar system III" by Hannes Alfvén, 1942, *Stockholm's Observatoriums Annaler*, v. 14, 9.1–9.29.

THE ANCIENT GREEKS knew the universe to be made up of the four elements: earth; water; wind; and fire. Today, we know these as the four states of matter: solid; liquid; gas; and plasma, three of which fall into the realm of *fluid dynamics*. Indeed, more than 99.9% of "ordinary matter" in the universe is in the fluid state and, in particular, the plasma (magnetohydrodynamical) state.[1]

Yet, as a pure science, fluid dynamics has often been omitted from many university undergraduate physics curricula. In fact, if you want to find regularly offered courses in fluid dynamics in a university calendar, you're more likely to find them among the engineering or applied mathematics offerings than physics.

One could come up with a number of reasons for this:

- areas of physics such as classical mechanics, electrodynamics, and quantum mechanics are deemed more "fundamental" and courses such as fluid dynamics get relegated as "optional", if offered at all;
- analytical progress generally requires mathematics not typically understood by most undergraduate students of physics until their fourth year; and
- historically, the really interesting problems required the use of major laboratory facilities (such as those available in a large engineering department) or theorems of advanced applied mathematics.

An alternative to expensive laboratories or a degree in Applied Mathematics is computing. While supercomputers capable of solving interesting problems in fluid dynamics have been available since the mid 1980s, it is only since the turn of the 21st century that *cheap* supercomputing has become widely available so that "ordinary" physicists and astrophysicists can once again do interesting problems in the subject.

Indeed, many of the more "interesting" problems in astrophysics such as those

[1] www.plasma-universe.com/99-999-plasma/.

in star formation, planetary discs, stellar evolution, the interstellar medium, formation of galaxies, galactic and extragalactic outflows and accretion, the early universe, cosmology, even the Big Bang itself have awaited this "promised land" of cheap supercomputing. Now that it has "arrived", more and more of the literature in astrophysics is being devoted to applications of fluid dynamics and, in particular, magnetohydrodynamics. More than for any other practitioner of physics, astrophysicists are finding the role of fluid dynamics is becoming *increasingly* important with time, not less. For this reason alone, I would argue, university physics curricula should be offering more courses in fluid dynamics, lest the discipline be taken over completely by the engineers and applied mathematicians!

Before we start, let us agree on some basic terms and their uses.

1. A *fluid* is a state of matter that can flow. A liquid is an *incompressible* fluid, while gas and plasma (ionised gas) are *compressible* fluids. A more technical definition of a fluid involves the notion of *granularity*, where the *mean free path* (or *collision length* defined as the distance a particle in the fluid can travel, on average, before colliding with another particle), δl, is much less than any measurable scale length of interest ($\mathcal{L}$). When $\delta l \ll \mathcal{L}$, a fluid can be treated as a *continuum* rather than as an *ensemble of particles* which simplifies the governing equations enormously.

2. *Fluid Dynamics*, a term which is interchangeable with *hydrodynamics* (HD), is the physics of fluid flow (compressible or incompressible), and involves the concepts of mass and energy conservation, Newton's second law, and an *equation of state*.

3. *Fluid Mechanics* has come to refer to fluid dynamics from an engineering vantage point, with more emphasis on experimentation than on theory. Typically (but not always), a text entitled *Fluid Mechanics* will be an engineering text, while a text entitled *Fluid Dynamics* will be a physics text. A notable exception is Landau and Lifshitz' classic text *Fluid Mechanics*, which, in many ways, is the definitive treatment of the subject from a theoretical physicist's perspective.

4. *Gas Dynamics* is compressible fluid dynamics in which all the fluid particles are neutral.

5. *Magnetohydrodynamics (MHD)* is compressible or incompressible fluid in which an appreciable fraction of the particles are charged (ionised) and where charge neutrality is observed at all length scales of interest. Thus, within any volume element however small, there must be as many negative charges as positive. In an MHD fluid, circulation of charged particles at the sub-fluid length scale implies a current and thus a magnetic field which, in turn, interacts with ionised particles on the post-fluid length scale. Note that an MHD fluid need not be 100% ionised for the equations of MHD to apply (*e.g.*, Chap. 10 on

non-ideal MHD). Neutrals in a partially (even a few percent) ionised fluid can couple to the magnetic field via collisions with charged particles. By contrast, a completely neutral gas can neither generate nor interact with a magnetic field.

An MHD fluid can be created from an HD fluid by increasing the ionisation fraction. For a gas, this can be done by increasing its temperature and thus compressible MHD fluids are plasmas. For a liquid such as water, the ionisation fraction can be increased by dissolving salts. While the earth's oceans permeated by the earth's magnetic field technically constitutes an MHD fluid, the weakness of the earth's magnetic field ($\sim 4 \times 10^{-5}$ T, $\beta \sim 10^9$ defined in §5.2)[2] and the extremely low fraction of particles that are ionised renders the MHD effects just about immeasurable.

6. *Plasma Physics* is the study of the collective behaviour of an ensemble of charged particles at length scales smaller than the fluid length scale thereby rendering the MHD equations inapplicable. Plasma physics is generally described by the *Vlasov–Boltzmann equation* which can account for non-fluid-like behaviour such as charge separation and plasma oscillations. An MHD fluid can be described as a plasma in which charge neutrality is observed at all length scales of interest, and thus MHD is an important special case of plasma physics. An excellent first text on plasma physics, which is beyond the scope of this text, is Volume 1 of Francis Chen's now-classic text *Plasma Physics and Controlled Fusion* (1984).

The equations of MHD reduce to the equations of HD when the magnetic induction ($\vec{B}$) is set to zero. As we shall see, HD becomes MHD by adding the Lorentz force to the hydrodynamic version of Newton's second law, and by introducing Faraday's law of induction that governs how the magnetic induction evolves. These modifications, which will seem rather elementary when first introduced, belie the incredible complexity magnetism provides an ionised fluid. For example, while a hydrodynamical fluid can support compressive waves only (and thus, much of HD can be understood in one dimension), the tension along lines of magnetic induction allow a magnetohydrodynamical fluid to support transverse waves as well, thus requiring all three dimensions to describe.

To understand MHD is to understand wave mechanics, and much of this text is devoted to building the students' mathematical skills and physical intuition in this area. By the end of Part I, the student will be able to solve the most complex MHD problem one can do exactly (albeit, semi-analytically), namely the MHD Riemann problem. And while the development of a general, multidimensional computer code

[2]Strictly speaking, it is the magnetic *induction*, $\vec{B}$, that has units tesla while the magnetic *field*, $\vec{H} = \vec{B}/\mu$ (App. B), has units ampere/metre. Thus, the earth's magnetic *field* is about 30 A/m. In this book, I attempt to be consistent with this distinction by using the term *magnetic field* when referring to magnetism generically, and *magnetic induction* when reference is to $\vec{B}$ specifically although, for the most part and especially in astrophysics, this difference is largely academic since all that separates them is the constant μ_0.

to solve more complex problems in MHD is beyond the scope of this book, the 1-D Riemann problem and the ideas upon which it is based are at the core of virtually every general computer program written and with which a whole host of interesting (astro)physical problems become accessible.

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PART I

1-D MHD IN TEN WEEKS

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1 The Fundamentals of Hydrodynamics

Everything flows and nothing abides; everything gives way and nothing stays fixed.

Heraclitus (c. 535–c. 475 BCE)

1.1 Definition of a fluid

THE PHYSICS of hydrodynamics (HD), namely conservation of mass, conservation of energy, and Newton's second law, are all concepts familiar to first-year undergraduate students, though the mathematics to solve the relevant equations is not. Consider an *ensemble of particles* within some volume V, and let these particles interact with each other via elastic collisions. We can let V remain fixed (in which case we allow the particles to collide elastically with the walls of the container too), or we can let V increase or decrease as the particles move apart or come together; it does not matter. If the mass, total energy, and momentum of the ensemble of particles are M, $E_{\rm T}$, and $\vec{S}$ respectively, then we have:

$$\frac{dM}{dt} = 0, \qquad \text{conservation of mass;} \tag{1.1}$$

$$\frac{dE_{\rm T}}{dt} = \sum \mathcal{P}_{\rm app}, \qquad \text{conservation of total energy;} \tag{1.2}$$

$$\frac{d\vec{S}}{dt} = \sum \vec{F}_{\rm ext}, \qquad \text{Newton's second law.} \tag{1.3}$$

Here, $\sum \mathcal{P}_{\rm app}$ is the rate at which work is done (power) by all forces *applied* to the ensemble of particles, and $\sum \vec{F}_{\rm ext}$ are all forces *external* to and acting on the ensemble of particles. Note that the applied forces – normally just collisions from neighbouring ensembles of particles – are typically a subset of the external forces, which include collisions from neighbouring particles *plus* forces arising from gravity, magnetism, radiation, *etc.* This is because in addition to the thermal and kinetic energies, the *total energy*, $E_{\rm T}$, includes gravitational, magnetic, radiative, and possibly other energies as well.

It is how we model the collisional forces from neighbouring ensembles of particles that defines both what constitutes a fluid and how Eq. (1.1)–(1.3) are further developed. Consider a small cube with volume $\Delta V = (\Delta l)^3$ as shown in Fig. 1.1*a*.

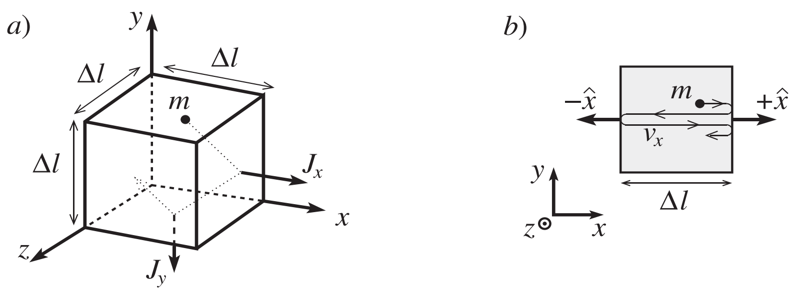

Figure 1.1. *a*) A single particle bounces elastically from the walls of a cube of edge length, Δl, imparting impulses J_x, J_y, *etc.* *b*) An x–y cut through the cube in panel *a* showing one particle whose motion is entirely in the x-direction.

Let the walls of the cube be perfectly reflecting and let there be just one particle inside the cube moving at some speed v in an arbitrary direction.

When the particle collides with the wall, both the particle and cube suffer a change in momentum in a direction normal to the surface of the cube. Moments later, the particle collides with a different wall, and the particle and cube suffer changes in momentum in a direction normal to that wall. A change in momentum is an impulse, J, which when multiplied by the time over which the collision occurs, Δt, constitutes the average force. Thus formally, the "pressure", p, the collision exerts on the wall of the box is this average force divided by the area of the wall:

$$p \sim \frac{J\Delta t}{(\Delta l)^2}.$$

In this scenario, the "pressure" is highly variable in time, and by no means could the "pressure" be construed as isotropic. At a given time, the "pressure" one wall feels will have nothing to do with the "pressures" felt by the other walls.

However, by arbitrarily increasing the number of particles, $\mathcal{N}$, inside our small volume, ΔV, the number of collisions with a given wall, n, occurring in a time Δt will be the same at each wall to within some arbitrarily small variance, Δn. Put another way, averaged over Δt, particle collisions exert the same "pressure" on each wall to within a variance made as small as we please by making $\mathcal{N}$ as large as we please. Thus, we have rendered the particle "pressure" inside the cube *isotropic* because each wall now feels the same force.

There is a contrived exception to this picture. If all the particles were to be placed initially on the mid-plane of the cube and all were launched with the same speed towards one wall of the cube, then it is only with this and the opposite wall that particles would ever collide, and they would do so in a highly ordered, periodic fashion. The remaining four walls would, in principle, never feel any collisions, and thus the "pressure" in the cube would not be isotropic even with $\mathcal{N}$ chosen arbitrarily large. Such a well-ordered and well-directed ensemble of particles is said to be *streaming* and, as $\mathcal{N}$ is made larger, it becomes increasingly difficult in practice to maintain streaming motion. Small perturbations will eventually cause one particle to collide with another which in turn collide with others, and the ensuing chain reaction quickly reduces the streaming motion to chaos. Isotropic "pressure" (the same "pressure" measured on each of the six walls) is once again the result.

We can now state the key criterion for an ensemble of particles to be treated as a fluid. If there is a sufficient number of particles inside our box (volume element) of dimension Δl so that the motion of particles within the volume element can always be considered isotropic, then the effect of the collisions of particles against the walls of the volume element (which may be rigid walls, or "soft" walls of neighbouring ensembles of particles) is to exert an isotropic "pressure" against all walls. Since isotropy is maintained by particle–particle collisions within the volume element, we may "mathematise" this criterion as,

$$\delta l \ll \Delta l < \mathcal{L}, \tag{1.4}$$

where δl is the mean free path (collision length) of the particles, Δl is the length of one side of our cubic volume element containing an arbitrarily large number of particles, and $\mathcal{L}$ is the smallest length scale of interest in our physical problem. If Ineq. (1.4) holds, we say the ensemble of particles behaves as a *fluid* or a *continuum*. This assumption is an important one; it allows us to treat the applied forces resulting from collisions – which otherwise could be *extremely* difficult to deal with – in a very simple way, namely as an isotropic "pressure".

1.2 A quick review of kinetic theory

To now, I have been enclosing the word *pressure* in quotation marks. This is because I haven't yet made the logical connection between particle collisions (and more specifically, the momentum transferred by particle collisions) and what we commonly think of as *pressure*, such as the *barometric* pressure of the air. So, before we examine how Eq. (1.1)–(1.3) become the equations of hydrodynamics (HD) under the assumption that the ensemble of particles behaves as a fluid (when Ineq. 1.4 is valid), let us review how the "pressure" and the "temperature" of a fluid relate to properties of the ensemble of particles. These ideas form the basis of *kinetic theory*, often exposed to students for the first time in a first-year physics course.[1]

Consider a cube whose edges of length Δl are aligned with the x-, y-, and z-axes of a Cartesian coordinate system, as depicted in Fig. 1.1. Returning to our example in the previous section, suppose a single point particle of mass m moves inside the cube with velocity $v_x\hat{x}$ and collides with the wall whose normal is $+\hat{x}$. If collisions are all elastic, then the particle reflects from the wall with a velocity $-v_x\hat{x}$ and thus suffers a change in momentum of $\Delta S_x = -2mv_x$. Conservation of momentum then demands that an impulse of $+2mv_x$ be imparted against the wall. At a time $\Delta t = 2\Delta l/v_x$ later, the same particle again collides with the wall, imparting another impulse of $+2mv_x$ against it. Thus, the rate at which momentum is delivered to the

[1] For example, Halliday, Resnick, & Walker (2003).

wall by a single particle is given by,

$$\frac{\Delta S_x}{\Delta t} = \frac{2mv_x}{2\Delta l/v_x} = \frac{mv_x^2}{\Delta l} = \langle F\rangle,$$

where $\langle F\rangle$ is the average force felt by the wall. Thus, the average pressure exerted by this one particle, defined as force per unit area, is given by,

$$\langle p\rangle = \frac{\langle F\rangle}{(\Delta l)^2} = \frac{mv_x^2}{V},$$

where $V = (\Delta l)^3$ is the volume of the cube. For $\mathcal{N}$ particles, we simply add over all particles:

$$p \equiv \sum_{i=1}^{\mathcal{N}}\langle p_i\rangle = \sum_{i=1}^{\mathcal{N}}\frac{mv_{x,i}^2}{V} = \frac{m}{V}\sum_{i=1}^{\mathcal{N}}v_{x,i}^2 = \frac{m\mathcal{N}}{V}\langle v_x^2\rangle, \tag{1.5}$$

where each point particle is assumed to have the same mass, m, and where $\langle v_x^2\rangle = \sum v_{x,i}^2/\mathcal{N}$ is the arithmetic mean of the squares of the particle velocities.

For any given particle, $v^2 = v_x^2 + v_y^2 + v_z^2$ and, for large $\mathcal{N}$, one would expect $\langle v_x^2\rangle = \langle v_y^2\rangle = \langle v_z^2\rangle$ since one Cartesian direction shouldn't be favoured over another. Thus,

$$\langle v^2\rangle = \langle v_x^2\rangle + \langle v_y^2\rangle + \langle v_z^2\rangle = 3\langle v_x^2\rangle, \tag{1.6}$$

and Eq. (1.5) becomes,

$$p = \frac{\mathcal{N}mv_{\rm rms}^2}{3V}, \tag{1.7}$$

where,

$$v_{\rm rms} \equiv \sqrt{\langle v^2\rangle},$$

is the *root-mean-square* (rms) speed of the particles in the volume V. Comparing Eq. (1.7) with the *ideal gas law*:

$$p = \frac{\mathcal{N}k_{\rm B}T}{V}, \tag{1.8}$$

(where $k_{\rm B} = 1.3807\times10^{-23}$ J K^{-1} is the Boltzmann constant) yields:

$$T = \frac{mv_{\rm rms}^2}{3k_{\rm B}} \quad\Rightarrow\quad \frac{3}{2}k_{\rm B}T = \frac{1}{2}mv_{\rm rms}^2 = \langle K\rangle, \tag{1.9}$$

where $\langle K\rangle$ is the average kinetic energy per point particle. Thus, while the *pressure*, p, is a measure of the rate at which momentum is transferred from the particles of the fluid (gas) to, for example, the diaphragm of the measuring device (barometer), the *temperature* (or more precisely $3k_{\rm B}T/2$) is a measure of the average kinetic energy of the particles.

The randomly directed kinetic energy of a system of $\mathcal{N}$ particles is called its *internal energy*, E, and, for the point particles under discussion, is given by,

$$E = \mathcal{N}\langle K\rangle = \frac{3}{2}\mathcal{N}k_{\rm B}T.$$

The factor 3/2 is significant and warrants comment. A point particle, as may be found exclusively in a monatomic gas, has three *degrees of freedom* of motion, namely *translation* in each of the three Cartesian directions (Fig. 1.2, left).

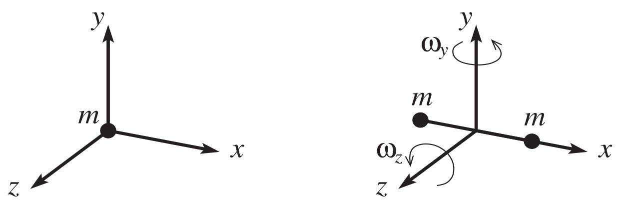

Figure 1.2. A point particle (left) has three degrees of freedom for movement, while a "dumb-bell" (right) has five.

From Eq. (1.6), we have $\langle v_i^2 \rangle = \langle v^2 \rangle/3$ for $i = x, y, z$, and thus to each (translational) degree of freedom we can associate an internal energy $E_i = \mathcal{N} k_B T/2$, where $E = E_x + E_y + E_z = 3\,E_i$.

Now, a diatomic molecule (essentially two point masses connected by a massless rod) has the same three translational degrees of freedom as a monatomic particle *plus* two *rotational* degrees of freedom, namely rotation about each of the two principle axes orthogonal to its own axis (the x-axis in Fig. 1.2, right), for a total of five degrees of freedom.[2] Note that spinning about the x-axis itself does not constitute a degree of freedom as the moment of inertia about this axis is essentially zero. Because of the *principle of equipartition*,[3] each degree of freedom stores the same amount of kinetic energy, and the internal energy of a diatomic gas must be,

$$E = \frac{5}{2}\mathcal{N} k_B T.$$

Thus, in general, we write,

$$E = \frac{1}{\gamma - 1}\mathcal{N} k_B T, \tag{1.10}$$

where $\gamma = 5/3$ for a monatomic gas, $\gamma = 7/5$ for a diatomic gas, and $4/3 \le \gamma < 7/5$ for molecules more complex than diatomic.[4] One can show that $\gamma = C_P/C_V$, the ratio of specific heats of the gas, and that for an adiabatic gas (where heat is neither lost nor gained from the system), $p \propto \rho^\gamma$, where ρ is the *mass density* of the gas.

Dividing Eq. (1.10) by the volume of the sample and using Eq. (1.8) gives an expression for the *internal energy density*, e:

$$e = \frac{E}{V} = \frac{1}{\gamma - 1}\frac{\mathcal{N} k_B T}{V} = \frac{p}{\gamma - 1}.$$

Thus, an alternate form of the ideal gas law, and the form most frequently used in

[2]In principle, there are also two vibrational degrees of freedom which, at "ordinary temperatures", statistical mechanics tells us are insignificant.

[3]Left to their own devices, systems will distribute the available energy equally among all possible ways energy can be stored. Thus, for a large number of diatomic molecules randomly colliding with each other and the walls of their container, one would not expect $m\langle v_x^2 \rangle$ to differ significantly from $m\langle v_y^2 \rangle$ or $m\langle v_z^2 \rangle$ any more than it should differ from $I_y\langle \omega_y^2 \rangle$ or $I_z\langle \omega_z^2 \rangle$, where I_y and I_z are the moments of inertia about the y- and z-axes respectively.

[4]Polyatomic molecules are significantly more complex than diatomic molecules, and the full power of statistical mechanics along with a tensor treatment of its moment of inertia are required to explain the value of γ for any individual molecule.