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Excerpt
More Information

Part I

Risk Bounds for Portfolios Based on Marginal Information

The basic assumption in Parts I–III of this book in order to derive risk bounds is that the marginal distributions of the risk vector $X = (X_1, \ldots, X_n)$ are known, say $X_i \sim F_i$, $1 \le i \le n$. This can be realistically assumed in many applications since it is much easier to model and test simple hypotheses compared to the task of modeling and testing the joint distribution of X. We often deal with risk bounds for the aggregated portfolio $S = \sum_{i=1}^{i=n}(X_i)$, but we also present risk bounds for other risk functionals, for example for the maximal risk, for the variation max $|X_i - X_j|$ of the risks, or for the risk exposure in certain domains.

In Chapter 1, we introduce some basic notions of risk measures (like VaR, TVaR, and convex risk measures) and describe some corresponding worst case VaR and TVaR portfolios. We also give a rearrangement formulation to determine worst or best case risk bounds. We describe the connection of upper and lower risk bounds to convex ordering properties, discuss comonotonicity and countermonotonicity (or antimonotonicity), and give some basic results to obtain worst case VaR portfolios resp. portfolios with maximal tail risk.

The "standard bounds" for tail risk go back to classical sources like Sklar (1973) or Moynihan et al. (1978). For $n = 2$ they are shown in Makarov (1981) and Rüschendorf (1982) to be sharp, but for $n > 2$ they typically only deliver rough bounds.

The conditional moment method gives an upper bound on the tail risk of the portfolio in terms of conditional moments of the marginals. This type of upper risk bound produces sharp bounds under a mixing condition on the upper tail. In the final subsection of this part, we discuss in more detail the notion of mixability, describe some basic results due to Wang and Wang (2011, 2016) on mixability, and explain its role in the determination of convex minima of portfolio sums and similarly for best and worst case portfolios.

Chapter 2 is devoted to the motivation and introduction of the rearrangement algorithm (RA) as introduced in Puccetti and Rüschendorf (2012a). This is a fundamental tool to determine sharp upper and lower bounds for the tail risk resp. for the VaR. It is basically motivated by the formulation of the problem of determining risk bounds as a rearrangement problem and in a second step by a further reduction to an assignment problem.

Chapter 3 gives an introduction to Hoeffding–Fréchet functionals, which describe the largest and the smallest value of a risk functional over all possible dependence structures, when fixing the marginals. The main result is a duality theorem for these functionals, which is a far reaching extension of the dual representation in the Monge–Kantorovich mass transportation problem. A reduction of the admissible dual functions to a simple class of admissible dual functions introduced in Embrechts and Puccetti (2006b) leads to good dual bounds for the tail risk. Sharpness of these bounds was established under a mixing condition in the homogeneous case in Puccetti and Rüschendorf (2013).

A simple upper bound for the worst case VaR risk is given by the TVaR bound, i.e., the TVaR of the comonotonic risk vector. In Chapter 4 we derive asymptotic sharpness of this upper TVaR bound under an asymptotic mixing condition. A version of this result also holds in the infinite mean case and in the inhomogeneous case.

1 Risk Bounds with Known Marginal Distributions

As described in the introduction, a key problem of risk analysis is to derive (sharp) risk bounds on a portfolio $S = X_1 + \cdots + X_n$ under the given distributional information on a risk vector $X = (X_1, \ldots, X_n)$. In this chapter, we derive several explicit results for this problem under the assumption that only the marginal distributions F_j of X_j are known, but the dependence structure of X is completely unknown. In particular we introduce some basic notions of risk theory, such as worst case value-at-risk and tail value-at-risk portfolios, comonotonic risk vectors, the connection of upper risk bounds to convex ordering, and some basic results to obtain worst case value-at-risk portfolios. A more detailed presentation and extension of these results is given in Rüschendorf (2013, Chapters 2–4). Some detailed mixing results in Section 1.4 are due to several papers of Wang and coauthors (see Wang and Wang, 2011).

1.1 Some Basic Notions and Results of Risk Analysis: VaR, TVaR, Comonotonicity, and Convex Order

There are several risk measures of interest, like the value-at-risk (VaR), the tail value-at-risk (TVaR), and the classes of convex risk measures or of distortion risk measures. The VaR risk measure at level α, VaR_α, $\alpha \in (0, 1)$ of the portfolio S is defined as the α-quantile of the distribution of S, i.e.,

$$\mathrm{VaR}_\alpha(S) = F_S^{-1}(\alpha) = \inf\{x \in \mathbb{R};\ \ F_S(x) \geq \alpha\}; \tag{1.1}$$

we also make use of the upper VaR as an upper α-quantile, i.e.,

$$\mathrm{VaR}_\alpha^+(S) = \sup\{x \in \mathbb{R};\ \ F_S(x) \leq \alpha\}. \tag{1.2}$$

The TVaR risk measure at level α, TVaR_α, takes into account also the magnitude of the risk above the α-quantile and is defined as

$$\mathrm{TVaR}_\alpha(S) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u^+(S)\,\mathrm{d}u = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(S)\,\mathrm{d}u. \tag{1.3}$$

From the definition it follows that TVaR is an upper bound of VaR, i.e., for $\alpha < 1$ it holds that

$$\mathrm{VaR}_\alpha(S) \leq \mathrm{VaR}_\alpha^+(S) \leq \mathrm{TVaR}_\alpha(S). \tag{1.4}$$

In comparison to VaR, TVaR has the important property of being a convex risk measure. A risk measure ϱ is said to be **convex** (Föllmer and Schied, 2004) if it is monotone, translation invariant, and satisfies the important convexity condition,

$$\mathrm{TVaR}_\alpha(aX + (1-a)Y) \le a\,\mathrm{TVaR}_\alpha(X) + (1-a)\,\mathrm{TVaR}_\alpha(Y). \tag{1.5}$$

If the risk measure ϱ is also positive homogeneous, then it is called **coherent**.

Thus, using TVaR as a risk measure, a diversified portfolio is preferred concerning the magnitude of risk in comparison to an undiversified portfolio. The left TVaR measure at level α, LTVaR_α is similarly defined and considers the left tails (best case) of risks:

$$\mathrm{LTVaR}_\alpha(S) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_u(S)\,\mathrm{d}s. \tag{1.6}$$

An important property of a risk measure that is convex **law invariant**, i.e., one that only depends on the marginal distribution, is its consistency with respect to convex order $\le_{\mathrm{cx}}$.

Definition 1.1 (Convex order) Let X and Y be two random variables with finite means. X is smaller than Y in convex order, denoted by $X \le_{\mathrm{cx}} Y$, if for all convex functions f,

$$Ef(X) \le Ef(Y), \tag{1.7}$$

whenever both sides of (1.7) are well defined.

A law-invariant convex risk measure ϱ (e.g., TVaR) is consistent with respect to convex order on proper probability spaces such as L^1 (integrable random variables) and L^∞ (bounded random variables). In consequence it holds that $X \le_{\mathrm{cx}} Y$ implies

$$\mathrm{TVaR}_\alpha(X) \le \mathrm{TVaR}_\alpha(Y), \tag{1.8}$$

see Chapter 4 of Föllmer and Schied (2004), Jouini et al. (2006), Bäuerle and Müller (2006), and Burgert and Rüschendorf (2006). From this section on, we consider as the basic space of risks $\mathcal{X} = L^1$ and assume that all marginal distributions of a risk vector X have finite first moments when dealing with TVaR. For given distribution functions $F_1, \dots, F_n$, let $\mathcal{F}(F_1, \dots, F_n)$ denote the **Fréchet class** of all n-dimensional distribution functions F with marginal distribution functions $F_1, \dots, F_n$. The classical Fréchet bounds characterize the Fréchet class $\mathcal{F}(F_1, \dots, F_n)$.

Theorem 1.2 (Fréchet bounds)
a) For $F \in \mathcal{F}(F_1, \dots, F_n)$ it holds that

$$\begin{aligned} F_-(x) &:= \Big(\sum_{i=1}^n F_i(x_i) - (n-1)\Big)_+ \le F(x) \\ &\le F_+(x) := \min_{1\le i\le n} F_i(x_i), \quad x \in \mathbb{R}^n. \end{aligned} \tag{1.9}$$

F_-, F_+ are called lower resp. upper Fréchet bounds (also called Hoeffding–Fréchet bounds).

b) $F_+ \in \mathcal{F}(F_1, \dots, F_n)$*; if* $n = 2$ *then* $F_- \in \mathcal{F}(F_1, F_2)$.

c) For a distribution function F *on* $\mathbb{R}^n$*, it holds that*

$$F \in \mathcal{F}(F_1, \dots, F_n) \Longleftrightarrow F_- \le F \le F_+.$$

In particular, there exists for any n a largest distribution function with marginals F_i, the upper Fréchet bound F_+. For $n = 2$ there exists a smallest distribution function with marginals F_i, the lower Fréchet bound. In general, the upper and lower bounds in (1.9) are sharp. The upper bound F_+ is attained by the comonotonic risk vector.

Definition 1.3 (Comonotonicity, countermonotonicity)
Let $F_1, \dots, F_n$ be one-dimensional distribution functions, and let $U \sim U(0,1)$ be uniformly distributed on $[0,1]$. Then:

a)
$$X^c := (F_1^{-1}(U), \dots, F_n^{-1}(U)) \tag{1.10}$$

with $F_i^{-1}(\alpha) = \inf\{x \in \mathbb{R} \colon F_i(x) \ge \alpha\}$ is called a **comonotonic** risk vector.

b) For $n = 2$,

$$X_c := (F_1^{-1}(U), F_2^{-1}(1 - U)) \tag{1.11}$$

is called a **countermonotonic** (antimonotonic) risk vector.

Comonotonic risk vectors X are characterized by the fact that the components of X are ordered in the same way.

The co- resp. countermonotonic risk vectors realize the upper resp. lower Fréchet bounds F_+, F_-, i.e.,

$$X^c \sim F_+ \qquad \text{and for } n = 2, \quad X_c \sim F_-. \tag{1.12}$$

In terms of the lower orthant order $\le_{\mathrm{lo}}$ defined by the pointwise ordering of the distribution functions, therefore, for any vector X with marginal distributions F_i it holds by the Fréchet bounds that

$$X \le_{\mathrm{lo}} X^c \tag{1.13}$$

and for $n = 2$,

$$X_c \le_{\mathrm{lo}} X. \tag{1.14}$$

The following basic result due to Meilijson and Nadas (1979) describes the role of the comonotonic vector as a worst case model for the portfolio $S = \sum_{i=1}^n X_i$ with respect to all law-invariant convex risk measures.

Theorem 1.4 (Comonotonic risk vector and convex order)
Let $X = (X_1, \dots, X_n)$ *be a risk vector with marginal distributions* F_i*. Then*

a)
$$\sum_{i=1}^n X_i \le_{\mathrm{cx}} \sum_{i=1}^n X_i^c, \tag{1.15}$$

i.e., the portfolio of comonotonic risks is the worst case portfolio with respect to convex order.

b)
$$E\Big(\sum_{i=1}^n X_i - t\Big)_+ \le E\Big(\sum_{i=1}^n X_i^c - t\Big)_+ \tag{1.16}$$

for all t. Moreover, $E\big(\sum_{i=1}^n X_i^c - t\big)_+ =: \Psi_+(t)$, *where*

$$\Psi_+(t) = \inf_{\sum v_i = t} \sum_{i=1}^n E(X_i - v_i)_+. \tag{1.17}$$

The statement in b) says that the excess of loss risk functional of the portfolio is maximized by the comonotonic risk vector.

For $n = 2$, a countermonotonic risk vector $X_c = (F_1^{-1}(U), F_2^{-1}(1-U))$ realizes the convex minimum of portfolio sums of variables X_i with distribution functions F_i.

Proposition 1.5 (Countermonotonic risk vector and convex order) *Let* $X = (X_1, X_2)$ *be a risk vector of size* $n = 2$ *with marginal distribution functions* F_i*. Then for all* $X_i \sim F_i$ *it holds that*

$$F_1^{-1}(U) + F_2^{-1}(1-U) \le_{\mathrm{cx}} X_1 + X_2. \tag{1.18}$$

In consequence, for $n = 2$ we have for all $X_i \sim F_i$,

$$X_{1,c} + X_{2,c} \le_{\mathrm{cx}} X_1 + X_2 \le_{\mathrm{cx}} X_1^c + X_2^c, \tag{1.19}$$

where $X_c = (X_{1,c}, X_{2,c})$.

We define the worst case risks of the portfolio $S = \sum_{i=1}^n X_i$, where X_i have marginal distribution functions F_i with respect to VaR and TVaR by

$$\begin{aligned} \overline{\mathrm{VaR}}_\alpha &:= \sup\Big\{\mathrm{VaR}_\alpha(S); S = \sum_{i=1}^n X_i, X_i \sim F_i, 1 \le i \le n\Big\} \\ \text{and} \quad \overline{\mathrm{TVaR}}_\alpha &:= \sup\Big\{\mathrm{TVaR}_\alpha(S); S = \sum_{i=1}^n X_i, X_i \sim F_i, 1 \le i \le n\Big\}. \end{aligned} \tag{1.20}$$

Similarly, the best case of risks at level α is defined as

$$\begin{aligned} \underline{\mathrm{VaR}}_\alpha &:= \inf\Big\{\mathrm{VaR}_\alpha(S); S = \sum_{i=1}^n X_i, X_i \sim F_i, 1 \le i \le n\Big\} \\ \text{and} \quad \underline{\mathrm{TVaR}}_\alpha &:= \inf\Big\{\mathrm{LTVaR}_\alpha(S); S = \sum_{i=1}^n X_i, X_i \sim F_i, 1 \le i \le n\Big\}. \end{aligned} \tag{1.21}$$

Then we get by means of Theorem 1.4 the following important connections between these notions. For a risk vector X, let $S = \sum_{i=1}^n X_i$ be the portfolio sum and $S^c = \sum_{i=1}^n X_i^c$ be the corresponding portfolio sum of the comonotonic risk vector X.

Theorem 1.6 *Let X be a risk vector with distribution function* $F \in \mathcal{F}(F_1, \ldots, F_n)$*. Then for the portfolio* $S = \sum_{i=1}^n X_i$*, it holds that*

a)
$$\mathrm{VaR}_\alpha(S) \le \mathrm{TVaR}_\alpha(S) \le \mathrm{TVaR}_\alpha(S^c) = \sum_{i=1}^n \mathrm{TVaR}_\alpha(X_i), \tag{1.22}$$

b)
$$\sum_{i=1}^n \mathrm{LTVaR}_\alpha(X_i) = \mathrm{LTVaR}_\alpha(S^c) \le \mathrm{LTVaR}_\alpha(S) \le \mathrm{VaR}_\alpha(S), \tag{1.23}$$

c)
$$\overline{\mathrm{VaR}}_\alpha \le \overline{\mathrm{TVaR}}_\alpha = \sum_{i=1}^n \mathrm{TVaR}_\alpha(X_i)$$
$$\textit{and} \quad \underline{\mathrm{LTVaR}}_\alpha = \sum_{i=1}^n \mathrm{LTVaR}_\alpha(X_i) \le \underline{\mathrm{VaR}}_\alpha, \tag{1.24}$$

d)
$$\mathrm{VaR}_\alpha(S^c) = \sum_{i=1}^n \mathrm{VaR}_\alpha(X_i). \tag{1.25}$$

Proof The inequality $\mathrm{VaR}_\alpha(S) \le \mathrm{TVaR}_\alpha(S)$ is immediate from the definition of $\mathrm{TVaR}_\alpha(S)$. Since TVaR_α is a convex law-invariant risk measure, we obtain the inequality $\mathrm{TVaR}_\alpha(S) \le \mathrm{TVaR}_\alpha(S^c)$ by the consistency with respect to convex order from Theorem 1.4.

Using that

$$\alpha\, \mathrm{LTVaR}_\alpha(S) + (1-\alpha)\, \mathrm{TVaR}_\alpha(S) = ES, \tag{1.26}$$

we obtain

$$\mathrm{LTVaR}_\alpha(S^c) = \sum_{i=1}^n \mathrm{LTVaR}_\alpha(X_i) \le \mathrm{LTVaR}_\alpha(S) \le \mathrm{VaR}_\alpha(S).$$

Finally for $S^c = \sum_{i=1}^n F_i^{-1}(U)$, it holds by the comonotonicity of the summands:

$$S^c \ge \mathrm{VaR}_\alpha(S^c)$$

if and only if for all i, $X_i = F_i^{-1}(U) \ge \mathrm{VaR}_\alpha(X_i)$, i.e.,

$$\mathrm{VaR}_\alpha(S^c) = \sum_{i=1}^n \mathrm{VaR}_\alpha(X_i), \tag{1.27}$$

$$\text{and} \quad \mathrm{TVaR}_\alpha(S^c) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(S^c)\,\mathrm{d}u \tag{1.28}$$
$$= \frac{1}{1-\alpha}\int_\alpha^1 \sum_{i=1}^n \mathrm{VaR}_u(X_i)\,\mathrm{d}u = \sum_{i=1}^n \mathrm{TVaR}_\alpha(X_i). \qquad \square$$

Remark 1.7 The inequalities (1.22) and (1.24) give a simple way to calculate an upper bound for the worst case VaR, whereas inequality (1.23) gives a lower bound for the best case VaR. The VaR of the comonotonic risk portfolio is easy to calculate, but it turns out that it is not a worst case with respect to VaR. The comonotonic portfolio is, however, a worst case portfolio with respect to TVaR, and hence the worst case TVaR bound is easy to determine. ◊

1.2 Standard Bounds, VaR Bounds, and Worst Case Distributions

It is an important task to describe good upper bounds for the value-at-risk and to determine worst case portfolios. The insight that the comonotonic portfolio is not the worst case VaR portfolio was a surprise in the practice of risk analysis and led to a rethinking of basic recommendations in risk regulation.

The standard bounds for the distribution function of the sum

$$M_n^{\leq}(t) = \sup\Big\{P\Big(\sum_{i=1}^n X_i \leq t\Big); X_i \sim F_i,\ \ 1 \leq i \leq n\Big\},$$

$$m_n^{\leq}(t) = \inf\Big\{P\Big(\sum_{i=1}^n X_i \leq t\Big); X_i \sim F_i,\ \ 1 \leq i \leq n\Big\},$$

resp. for the corresponding tail risks

$$M_n(t) = \sup\Big\{P\Big(\sum_{i=1}^n X_i \geq t\Big); X_i \sim F_i,\ \ 1 \leq i \leq n\Big\},$$

$$m_n(t) = \inf\Big\{P\Big(\sum_{i=1}^n X_i \geq t\Big); X_i \sim F_i,\ \ 1 \leq i \leq n\Big\},$$

have been known in the literature for a long time, see Sklar (1973), Moynihan et al. (1978), Denuit et al. (1999), and Rüschendorf (2005).

Theorem 1.8 (Standard bounds) *Let $X = (X_1, \dots, X_n)$ be a random vector with marginal distribution functions $F_1, \dots, F_n$. Then for any $t \in \mathbb{R}$, it holds that*

$$\begin{aligned}\Big(\bigvee_{i=1}^n F_i(t) - (n-1)\Big)_+ &\leq P\Big(\sum_{i=1}^n X_i \leq t\Big)\\ &\leq \min\Big(\bigwedge_{i=1}^n F_i(t), 1\Big),\end{aligned} \tag{1.29}$$

where $\bigwedge_{i=1}^n F_i(t) = \inf\{\sum_{i=1}^n F_i(u_i); \sum_{i=1}^n u_i = t\}$ is the "infimal convolution" of the (F_i), and $\bigvee_{i=1}^n F_i(t) = \sup\{\sum_{i=1}^n F_i(u_i); \sum_{i=1}^n u_i = t\}$ is the "supremal convolution" of the (F_i).

Proof For any $u_1, \dots, u_n$ with $\sum_{i=1}^n u_i = t$, it holds that

$$\begin{aligned}P\Big(\sum_{i=1}^n X_i \leq t\Big) &\geq P\Big(\bigcup_{i=1}^n (X_i \leq u_i)\Big),\\ &\geq \sum_{i=1}^n F_i(u_i),\end{aligned} \tag{1.30}$$

which implies the upper bound in (1.29). Similarly, using the Fréchet lower bound in (1.9) we obtain

$$\begin{aligned}P\Big(\sum_{i=1}^n X_i \leq t\Big) &\geq P\Big(X_1 \leq u_1, \dots, X_n \leq u_n\Big)\\ &\geq \Big(\sum_{i=1}^n F_i(u_i) - (n-1)\Big)_+.\end{aligned} \tag{1.31}$$

□

In general, the standard bounds in Theorem 1.8 are not sharp and can be considerably improved. Define for general n,

$$A_n(t) := \Big\{(x_1, \dots, x_n);\ \sum_{i=1}^n x_i \le t\Big\},$$
$$A_n^+(t) := \Big\{(x_1, \dots, x_n);\ \sum_{i=1}^n x_i < t\Big\}, \quad t \in \mathbb{R}^1,$$

and let

$$(F_1 \wedge F_2)^-(t) = \inf\{F_1(x-) + F_2(t-x); \quad x \in \mathbb{R}^1\}$$

denote the left continuous version of $F_1 \wedge F_2$; similarly, let $(F_1 \vee F_2)^-(t)$ be the left continuous version of $F_1 \vee F_2$. In the case $n = 2$, it was proved independently in Makarov (1981) and Rüschendorf (1982) that the standard bounds are sharp.

Theorem 1.9 (Sharpness of standard bounds, $n = 2$) *If X_i have distribution functions F_i, $1 = 1, 2$, then*

$$P(X_1 + X_2 \le t) \le M_2^{\le}(t) = (F_1 \wedge F_2)^-(t) \tag{1.32}$$

and

$$P(X_1 + X_2 < t) \ge m_2^{<}(t) = ((F_1 \vee F_2)^-(t) - 1)_+. \tag{1.33}$$

The proof of Theorem 1.9 given in Makarov (1981) uses direct arguments on the copulas, while the proof in Rüschendorf (1982) is based on duality theory. This latter proof allows us also to determine the worst case dependence structure.

On the unit interval [0, 1] supplied with the Lebesgue measure λ, define the random variables

$$Y_1(s) = F_1^{-1}(s), \quad Y_2(s) = F_2^{-1}(\varphi(S)), \tag{1.34}$$

with $\varphi(s) = 1 - s$, $0 \le s \le h(t)$, and $\varphi(s) = s$, $h(t) \le s \le 1$. Then the random variables Y_1, Y_2 maximize the distribution function of the sum at point t, i.e., they maximize $P(X_1 + X_2 < t)$. This means that they minimize the tail risk $P(X_1 + X_2 \ge t)$.

Proposition 1.10 (Maximizing (best case) pairs) *The random variables defined in* (1.34) *satisfy:*

a) $$Y_1 \sim F_1, \quad Y_2 \sim F_2,$$

b) $$P(Y_1 + Y_2 \le t) = M_2^{\le}(t) = (F_1 \wedge F_2)^-(t). \tag{1.35}$$

Proof The Lebesgue measure λ is invariant with respect to φ, i.e., $\lambda^\varphi = \lambda$. Therefore, $\lambda^{Y_i} = \lambda^{f_i^{-1} \circ \varphi} = \lambda^{F_i^{-1}}$, and thus $Y_i \sim F_i$, $i = 1, 2$. Since $F_i^{-1} \circ F_i(x) \le x$, we obtain for $s = F_1(u)$,

$$\begin{aligned} F_1^{-1}(s) + F_2^{-1}(h(t) - s) &= F_1^{-1} \circ F_1(u) + F_2^{-1}(h(t) - F_1(u)) \\ &= u + F_2^{-1}(F_2(t-u)) \le u + (t-u) = t. \end{aligned}$$