Fixed Point Theory and Variational Principles in Metric Spaces

The fixed point theory in metric spaces came into the existence through the PhD work of Polish mathematician Stefan Banach in 1920. The outcome of the Banach contraction principle became the initial source of the theory. It evolved with time and is now important not only for nonlinear analysis but also for many other branches of mathematics. It has also been applied to sciences and engineering. Many extensions and generalizations of the Banach contraction principle are explored by mathematicians. The proposed book covers some of the main extensions and generalizations of the principle. It focuses on the basic techniques and results of topics like set-valued analysis, variational principles, and equilibrium problems. This book will be useful for researchers working in nonlinear analysis and optimization and can be a reference book for graduate and undergraduate students.

There are some excellent books available on metric fixed point theory, but the above-mentioned topics are not covered in any single resource. The book includes a brief introduction to set-valued analysis with a focus on continuity and the fixed-point theory of set-valued maps and the last part of the book focuses on the application of fixed point theory.

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To our families

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Preface

The fixed point theory is an important tool not only of nonlinear analysis, but also for many other branches of modern mathematics. It has numerous applications within mathematics and has been applied in diverse fields such as medical sciences, chemistry, economics, management, engineering, game theory, and physics.

Historically the beginning of metric fixed point theory goes back two centuries, but its name was coined only in 1922 after the pioneer work of Polish mathematician Stefan Banach in his Ph.D. dissertation. Many remarkable results of fixed point theory have been obtained during nineteen sixtees and nineteen seventies such as Caristi's fixed point theorem, Nadlar's fixed point theorem, etc. A large number of research papers have already appeared in the literature on extensions and generalizations of the Banach contraction principle.

On the other hand, Ivar Ekeland established a result on the existence of an approximate minimizer of a bounded below and lower semicontinuous function in 1972. Such a result is now known as Ekeland's variational principle. It is one of the most elegant and applicable results that appeared in the area of nonlinear analysis with diverse applications in fixed point theory, optimization, optimal control theory, game theory, nonlinear equations, dynamical systems, etc. Later, it was found that several well-known results, namely, the Caristi–Kirk fixed point theorem, Takahashi's minimization theorem, the Petal theorem, and the Daneš drop theorem, from nonlinear analysis are equivalent to Ekeland's variational principle in the sense that one can be derived by using the other results.

The set-valued maps, also called multivalued maps or point-to-set maps or multifunctions, are first considered in the famous book on topology by Kuratowski. Other eminent mathematicians, namely, Painlevé, Hausdorff, and Bouligand, have also visualized the vital role of set-valued maps as one often encounters such objects in concrete and real-life problems.

During the last decade of the last century, the theory on equilibrium problems emerged as one of the popular and hot topics in nonlinear analysis, optimization, optimal control, game theory, mathematical economics, etc. The equilibrium problem is a unified model of several fundamental mathematical problems, namely, optimization problems, saddle point problems, fixed point problems, minimax inequality problems, Nash equilibrium problem, complementarity problems, variational inequality problems, etc. In 1955, Nikaido and Isoda first considered the equilibrium problem as an auxiliary problem to establish the existence results for Nash's equilibrium points in noncooperative games. In the theory of equilibrium problems, the key contribution was made by Ky Fan in 1972, whose new existence results contained the original techniques that became a basis for most further existence theorems in the setting of topological vector spaces. That is why equilibrium problem is also known as the Ky Fan type inequality.

There are some excellent books available on metric fixed point theory. However, most of them are inaccessible for the systematic study of fixed point theory, set-valued analysis, variational principles, and equilibrium problems on one platform. The main purpose of this book is to present the basic techniques and results of these topics. The idea of writing this book came into existence while the first author was teaching a course of Nonlinear Functional Analysis at the master's level from where Chapters 2, 3, and 4 were conceived.

The present book contains six chapters.

The first chapter is devoted to the basic definitions, examples, and results concerning metric spaces that are essential for rest of the book.

In Chapter 2, the Banach contraction principle and several of its generalizations, namely, Boyd–Wong fixed point theorem for ψ-contraction mappings, fixed point theorem for weakly contraction mappings, etc., are discussed. The completeness of the metric space under the condition of the Banach contraction principle is included. Caristi's fixed point theorem and some of its consequences and generalizations are also presented.

In Chapter 3, we first give an elementary treatment of set-valued maps and then provide a complete discussion on continuity of set-valued maps. The Hausdorff metric on the family of nonempty closed bounded subsets of a metric space is presented along with the continuity of set-valued maps in terms of the Hausdorff metric. The fixed point theory for set-valued maps is also discussed. In particular, Nadler's theorem, fixed point theorem for directional contraction set-valued maps, Caristi–Kirk fixed point theorem, fixed point theorem for dissipative set-valued maps, Mizoguchi–Takahashi fixed point theorem for Ψ-contraction set-valued maps, and fixed point theorem for weakly contraction set-valued maps are presented. The stationary points for set-valued maps and DMH theorem and its consequences are also given.

In Chapter 4, we present several forms of Ekeland's variational principle and their generalizations with applications to fixed point theory and optimization. The Borwein–Preiss variational principle and Takahashi's minimization principle are also discussed with their applications to fixed point theory and weak sharp minima.

Chapter 5 provides an elementary treatment of the theory of equilibrium problems and the equilibrium version of Ekeland's variational principle, also known as Ekeland's variational principle for bifunctions or extended Ekeland's variational principle. Several equivalent results of extended Ekeland's variational principle, namely, extended Takahashi's minimization theorem, Caristi–Kirk fixed point theorem, and Oettli–Théra theorem, are also presented. The concept of weak sharp solutions for equilibrium problems is discussed.

The last chapter discusses several applications of fixed point results to the system of linear equations, differential equations and delay differential equations, second order two-point boundary value problems, and various kinds of integral equations.

The book is prepared for the graduate and undergraduate students and can also be useful for the researchers working in the area of nonlinear analysis and optimization.

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Notations and Abbreviations

$\varnothing$	the empty set
$\|\|x\|\|$	the norm of the vector x in $\mathbb{R}^n$
$\langle x, y\rangle$	the scalar product of x and y
$B[a, b]$	the space of all bounded real-valued functions defined on $[a, b]$
$C[a, b]$	the space of all continuous real-valued functions defined on $[a, b]$
$P[a, b]$	the space all polynomials defined on $[a, b]$
ℓ^∞	the space of all bounded sequences of real or complex numbers
ℓ^p	the space of all sequences $\{x_n\}$ of real or complex numbers such that $\sum_{n=1}^{\infty} \|x_n\|^p < \infty$ for all $p \geq 1$
A°	the interior of the set A
int$A$	the interior of the set A
bd(A)	the boundary of the set A
$\overline{A}$	the closure of the set A
clA	the closure of the set A
$\mathrm{cl}_X\mathrm{A}$	the closure of the set A in X
A^c	the complement of the set A
$X \setminus A$	the complement of the set A in X
A'	the derived set of A
diam(A)	the diameter of the set A
c	the space of all convergent sequences of real or complex numbers
s	the space of all sequences of real or complex numbers
$S_r(x)$	the open sphere (or open ball) with center at x and radius r
$S_r[x]$	the closed sphere (or closed ball) with center at x and radius r

$\mathbb{C}$	the set of all complex numbers
$\mathbb{C}^n$	the space of ordered n-tuples complex numbers
$\mathbb{N}$	the set of all natural numbers
$\mathbb{Q}$	the set of all rational numbers
$\mathbb{Z}$	the set of all integers
$\mathbb{R}$	the set of all real numbers
$\overline{\mathbb{R}}$	the extended real line
$\mathbb{R}_+$	the set of all nonnegative real numbers
$\mathbb{R}^n$	the n-dimensional Euclidean space
2^X	the family of all subsets of X
2^X_{cl}	the family of all nonempty closed and bounded subsets of X
2^X_b	the family of all nonempty bounded subsets of X
2^X_q	the family of all nonempty compact subsets of X
sup	supremum
lim sup	limit supremum
inf	infimum
lim inf	limit infimum
dom(f)	the domain of a single-valued map f
Dom(F)	the domain of a set-valued map F
graph(f)	the graph of a single-valued map f
Graph(F)	the graph of a set-valued map F
Image(F)	the image of a set-valued map F
EVP	Ekeland's variational principle
CMP	constrained minimization problem
MP	minimization problem