

Notation Index

- 01-sequence, 147, 406
 \doteq , 204, 270
 A_w , 152
 A_α , 294
 $\mathbb{B}G$, 1
 $C(w)$, 270
 $c_\alpha(v, w)$, 354
 $c'_{\lambda\mu}$, 137
 c_{uv}^w , 169, 353
 $D_w(F_\bullet \cap E^\bullet)$, 200
 $D_w^\infty(F_\bullet \cap E^\bullet)$, 219
 D_r , 185
 $D_r(\varphi)$, 195
 $D_w(F_\bullet \rightarrow E_\bullet)$, 199
 D_J , 290
 D_α , 291
 D_v , 292
 \mathbb{D}_w^∞ , 219
 \mathbb{D}_r , 195
 \mathbb{D}_r^G , 198
 ∂_α , 299
 Δ_P , 274
 $\Delta_\lambda(c)$, 42
 $\mathbb{E}G$, 1
 $\varepsilon_p^T(X)$, 322
 $F(W, Q)$, 294
 $F(W, \Lambda)$, 294
 $\mathbf{Fl}(\mathbf{d}, V)$, 46
 Fl^∞ , 213
 $\text{Fr}(E)$, 9
 $\mathbf{Fl}_\omega(V)$, 230
 $Fl_\omega(V)$, 230
 Fl_ω^∞ , 237
 Γ , 232, 396
 $\Gamma^{(n)}$, 233, 396
 \mathfrak{g}_β , 267
 $\mathbf{Gr}(d, V)$, 44
 Gr^∞ , 211
 $H_G^*(X)$, 1
 $H_G^*(X, Y)$, 34
 $\eta_p X$, 320
 $h_p(x|y)$, 134
 \mathbf{k}_w , 152
 $LG(V)$, 230
 LG^∞ , 237

434

Notation Index

- | | |
|--------------------------------------|--------------------------------|
| Λ_G , 2 | σ_w , 163 |
| Λ_{rt} , 357 | $\tilde{\sigma}_\lambda$, 136 |
| \mathcal{L}_λ , 285 | $\tilde{\sigma}_w$, 163 |
| $\ell(w)$, 159, 241, 268 | s_β , 267 |
| M_{rt} , 267 | $s_\lambda(x y)$, 133 |
| M_{wt} , 267 | $s_\lambda(x)$, 40 |
| n_β^α , 269 | $S_w(c; x; y)$, 216 |
| Ω_I , 128 | $S_w^C(c; x; y)$, 256 |
| Ω_I° , 126 | τ_w , 303 |
| Ω_λ , 128 | U_P , 274 |
| Ω_λ° , 126 | U_β , 267 |
| $\tilde{\Omega}_\lambda$, 135 | U_σ , 114 |
| $\tilde{\Omega}_\lambda^\circ$, 135 | $U^-(w)$, 156 |
| Ω_w^∞ , 214 | $U(w)$, 270 |
| Ω_w , 153 | $V(\tau)$, 114 |
| Ω_w° , 153 | V_λ , 43 |
| $\Omega_{[w]}$, 171 | W_P , 275 |
| $PP^*(\Sigma)$, 122 | \dot{w} , 268 |
| Φ^w , 302 | \hat{w} , 186 |
| φ_v , 340 | $w(\underline{\alpha})$, 329 |
| ψ_v , 297 | w^{\max} , 171, 275 |
| p_I , 127, 249 | w^{\min} , 171, 275 |
| p_λ , 127, 243 | w_\circ , 159 |
| p_w , 154, 244, 271 | w_\circ^P , 275 |
| $p_{[w]}$, 275 | $X(\Sigma)$, 113 |
| p_v , 327 | $X(\underline{\alpha})$, 327 |
| p_v^w , 302 | $X(w)^\circ$, 271 |
| $Q_\lambda(c)$, 233, 394 | $X[w]^\circ$, 276 |
| $Q_\lambda(q y)$, 398 | $x(I)$, 328 |
| R^+ , 266 | $x(w)$, 273 |
| R_P^+ , 274 | $Y(w)^\circ$, 271 |
| \mathbf{r}_w , 152 | $Y[w]^\circ$, 276 |
| $S_m^{(n)}$, 174 | $y(I)$, 333 |
| $\mathfrak{S}_w(x; y)$, 7, 165 | $y(w)$, 273 |
| $\text{Symm}(x)$, 213 | $Z(\underline{\alpha})$, 326 |
| σ_λ , 136 | $Z(v)$, 292 |

Subject Index

- additive group
 - cohomology of, 22
 - fixed point of, 65
- adjoint group, 268, 357
- AJSB formula, 339–342, 348
- approximation space, xii, 3,
 - 12–16, 24, 197, 311,
 - 360–362, 416
- Atiyah–Bott formula, 77
- Atiyah–Bredon sequence, 111,
 - 124
- balanced product, 10, 12
- Berline–Vergne formula, 77
- Białynicki–Birula
 - decomposition, 75, 78, 97
- bialternant, 40, 133, 238
- bilinear form
 - symplectic, 228
 - W -invariant, 269
- Billey’s formula, 339–342, 348
- blowup
 - cohomology of, 80–83
 - fixed points, 82
 - tangent space, 82
- Borel
 - construction, xi, 1, 3
 - presentation, 283–288, 339
 - seminar on transformation
 - groups, xi, 1, 8, 77
 - subgroup, 21, 57, 153, 265
- Borel–Moore homology,
 - 373–374
- Bott–Samelson variety, 326–339
 - cell decomposition, 328
 - cohomology of, 337–339
 - desingularizes Schubert
 - variety, 330–331
 - fixed points, 327–328
 - invariant curves, 331
 - tangent space, 328–329
- braid relation, 166
- Bruhat decomposition, 154,
 - 188, 270, 273, 275
- Bruhat order, 158–161, 179,
 - 247–248, 272
- covering relation, 350

436

Subject Index

- induction on, 165
- subword criterion, 160, 272, 339, 349
- tableau criterion, 158
- BSDH variety, 348, *see also* Bott–Samelson variety
- cap product, 371
- Cauchy formula
 - for Schubert polynomials, 204–206, 223, 263
 - for Schur functions, 56, 143
- cell decomposition, 57, 314, 376–377
 - Białynicki–Birula theorem, 75
- chain, 327
- Chang–Skjelbred theorem, 93, 110
- character
 - direction of, 99
 - of Borel group, 22
 - of invariant curve, 98
 - of torus, 26
 - parallel, 99, 349
 - primitive, 99
- character group, 26
- characteristic class, 415–419
- Chern class, 16, 379–381
 - equivariant, 16–18
 - of representation, 18
 - properties of, 17, 379–381
- Chevalley formula, 350–353, 368
 - for G/B , 351
 - for G/P , 353
 - for Grassmannian, 138–140
- Chevalley–Monk formula, 170, 180
- Chevalley–Pieri formula, 138–140, 150
- Chinese remainder theorem, 66
- Chow envelope, 417
- Chow group, 377
 - equivariant, 314
 - of classifying space, 24
- circle group, 2
- classifying space, xi
 - as base of Borel construction, 2
 - as base of universal principal bundle, 12
- co-isotropic subspace, 229
- cohomology, 370–373
 - bivariant, 388
 - Čech–Alexander–Spanier, 383
 - compact support, 372
 - reduced, 372
- compact group, 63, 281–283, 334
 - complexification, 281
- complete conics, 80, 83–90
 - as blowup, 83–84
 - fixed points, 86–88
 - moduli space, 84
 - tangent weights, 87
- complex analytic space, 374
 - quotient by complex group, 24
- cone
 - Spec of graded algebra, 315
 - convex, 113

Subject Index

437

- dual, 113
- normal, 317
- tangent, 319
- convolution, 305–309
 - and divided difference operators, 306
 - of Schubert classes, 308
- coroot, 267
- coset representative, 171, 174, 275
- cup product, 370
- deformed group ring, 123, 124
- degeneracy locus, 7, 182–207
 - Cohen–Macaulay, 191, 196, 206
 - equivariant class, 129, 189–193, 198
 - for symmetric map, 226–227, 252–254
 - for vector bundle map, 194
 - irreducibility, 186–189
 - Lagrangian, 253, 259
 - universal, 197
- degeneration, 365–367
- Dehn–Sommerville relations, 122
- Demazure product, 330
- desingularization, 329–333
- determinantal ideal, 194, 195
- diagonal embedding, 38, 368
- diagonal trick, 220, 222, 259
- diagonalizable group, 67
 - fixed points of, 67, 77, 94, 105
- diagram
 - of permutation, 193–194, 206, 215
 - of signed permutation, 262
- dimension function, 152, 245
- divided difference operator, 165–168, 179, 220, 255–256, 290–309
 - action on Schubert classes, 291, 304
 - compatible with localization, 294–299
 - independent of reduced word, 166, 292, 300
 - left- and right-handed, 303, 306
 - Leibniz formula, 166, 294, 296, 299
 - via Bott–Samelson variety, 332
 - via correspondence, 292–293
- dual curve, 83
- duality
 - between Schur determinants, 42
 - Grassmann, 45, 141–144, 203–204
 - of Schubert classes in G/B , 343–344, 346, 354
 - of Schubert polynomials, 192, 201–204, 223, 263
- effective
 - class, 358
- effective class, 359, 362, 365–367
- equivariant cohomology, xi

438

Subject Index

- definition, 12
- for free action, 32–33
- for trivial action, 32
- functoriality, 14, 27–29
- homotopy invariance, 29–32
- equivariant formality, xi, 75–77, 110
- equivariant Hilbert polynomial, 324
- equivariant line bundle, 26, 285–286, 299–301
 - and divided difference operator, 300
 - and Schubert divisor in G/B , 353
- equivariant map, 28
- equivariant multiplicity, 320–324
 - of Schubert variety, 345–347
 - properties, 322–323
- equivariant vector bundle, 16–17
- essential set, 193–194, 206, 215, 262
- excision, 371
- fan, 113
 - normal to polytope, 114
 - simplicial, 124
- fiber bundle, 3, 377–379, 415
 - in Borel construction, 3–5
 - in localization, 5
 - Poincaré duality, 37–38
- finite cover, 36
- finite group
 - cohomology of, 23
 - five conics, 79
- fixed point
 - attractive, 96–97
 - isolated, 64, 96, 99
 - nondegenerate, 319
 - restriction to, 4, 5, 62
 - singular, 100
- flag bundle, 10, 44–47, 199
 - as tower of Grassmann bundles, 46, 230
 - complete, 46
 - isotropic, 230–232
 - partial, 46
- flag variety, 7, 153–180
 - as approximation space, 21
 - cell decomposition, 154
 - cohomology of, 48
 - fixed points, 154–156
 - generalized, 270
 - tangent weights, 155
 - tautological bundles, 46
- forgetful homomorphism, 4
- four lines, 72–74
- frame bundle, 9
- free action, 32
- fundamental class, 16, 372, 374–377
 - equivariant, 16–18, 313
 - for limit of subvarieties, 385
 - in Borel–Moore homology, 313
 - properties of, 17
 - pullback, 375
 - pushforward, 382
- fundamental weight, 267, 298

- G/B , 270–274
 cohomology of, 286–288
 fixed points, 271
- G/P , 274–279
 fixed points, 275
- general linear group
 cohomology of, 18–19
- Giambelli formula, 129, 150,
 164, 182–184, 206, 227, 238
- GKM theorem, 93
- GKM variety, 107
- graded-commutative ring, 370,
 416
- graph, 183, 190, 200, 227, 234,
 253
- graph Schubert variety, 222
- Grassmann bundle, 10, 44–46
 tautological bundles, 11
- Grassmann duality, 45
- Grassmannian, 126–151
 as approximation space, 19
 cell decomposition, 126–127
 cohomology of, 19, 50–54
 fixed points, 66–68
 tangent bundle, 45
 tangent weights, 67
 tautological bundles, 45
- group cohomology, 23
- Gysin homomorphism, 381–382
 equivariant, 34–37
 for Borel–Moore homology,
 312
 in localization theorem, 66,
 70, 77
- Gysin sequence, 25, 95
- Hilbert series, 121
- Hom bundle, 196
- homogeneous space, 6,
 265–288, 357, 358
 cohomology of, 32
- homology, 370–373
 Borel–Moore, 373–374
 equivariant, xiv, 311–315
- Hurewicz isomorphism, 383
- infinite flag variety, 208–224
 cohomology of, 210, 213
- infinite Grassmannian, 211–224
 cohomology of, 211–213
- infinite isotropic flag variety,
 237
- infinite Lagrangian
 Grassmannian, 237
- integration formula, 5, 69–74,
 77
 computes divided difference
 operator, 295
 computes structure
 constants, 352
 implies divisibility condition,
 72, 108
 in enumerative geometry,
 88–91
- interpolation
 characterizing double Schur
 polynomial, 134, 150
 characterizing Schubert
 polynomials, 222–223
 determining cohomology
 class, 132, 150

440

Subject Index

- intersection multiplicity, 358, 376
- invariant curve, 97–100
 character of, 98, 107
 criterion for finitely many, 99
 in G/P , 279–281, 295
 in Bott–Samelson variety, 331
 in flag variety, 157–158
 in Schubert variety, 280, 345
- inversion, 159, 268
- irreducible factor, 103, 105
- isotropic flag, 228
- isotropic flag variety, 230–232
 as flag bundle over
 Lagrangian
 Grassmannian, 234
 cohomology of, 231–232, 236–237
 fixed points, 244–245
 tangent weights, 245
 tautological bundle, 231
- isotropic subspace, 227–230
- Jacobi–Trudi determinant, 41–42, 133, 135
- Joseph polynomial, 324
- Jouanolou’s trick, 417
- Kempf–Laksov formula, 129–131, 135, 137, 150, 184, 220
 dual, 142
- Kleiman–Bertini transversality, 6, 357–358, 368
- Kronecker isomorphism, 372
- Künneth theorem, 32, 371, 378
 decomposition of the
 diagonal, 38–39, 55, 205
- Lagrangian Grassmannian, 230, 233–236
 cohomology of, 234–237
 fixed points, 242–244
 tangent weights, 243
- Lagrangian subspace, 228
- Lefschetz trace formula, 77
- Leibniz formula, 166, 256, 294, 296, 299
- length
 of partial permutation, 187
 of permutation, 159
 of signed permutation, 241
 of strict partition, 241
 of Weyl group element, 268
- Leray–Hirsch theorem, 377–379
 for comparing approximation
 spaces, 13
 for equivariantly formal
 spaces, 75
- linear algebraic group, 265–270, 288
 semisimple, 265
- Littlewood–Richardson
 coefficient, 137, 145
- Littlewood–Richardson rule, 145–149
 equivariant, 146–150
- local class, 320
- localization, 4–5, 62–78, 93–111
 Chang–Skjelbred theorem, 103–104

Subject Index

441

- divisibility condition,
 - 100–102, 108, 110, 132
- for Borel–Moore homology,
 - 318–320
- for Bott–Samelson variety,
 - 333–337
- for nonsingular varieties,
 - 65–66, 68–69
- for Schubert variety, 339–343
- general theorem, 93–96, 103, 110
- GKM theorem, 106, 109, 113
- image theorem, 100–110
- integration formula, 5, 69–74
- long exact sequence, 34, 313, 371, 373
- longest element, 159, 241, 268, 275
- matching, 392
- Mayer–Vietoris sequence, 33, 371
- Milnor construction, 23, 25
- Molev–Sagan theorem, 140
- moment graph, 72, 91, 107–108
 - convexity, 110
- moment map, 110
- Monk formula, 170, 180
- multidegree, 324
- multiplicative group
 - cohomology of, 2, 15
- Nägelsbach–Kostka formula, 41
- nil-Coxeter algebra, 309
- nil-Hecke algebra, 292, 307–309
- normal cone, 317
 - deformation to, 321, 388
 - specialization to, 318, 389
- one-line notation, 152, 241
- opposite Borel subgroup, 266
- opposite flag, 135, 229
- orientation class, 372, 390
- orthogonal group, 22, 24
- orthogonal space, 228
- parabolic subgroup, 274–279
 - cohomology of, 31
 - maximal, 274, 278
 - minimal, 274, 279, 291
 - stabilizer of point in
 - Grassmannian, 53
- partial flag variety, 171–173, 274–279
- partition, 40
 - complementary, 55
 - conjugate, 41, 184, 204
 - dual, 135, 242
 - strict, 232, 240–242, 396, 400
- permutation, 152
 - Grassmannian, 178, 204, 279
 - matrix, 152, 187
 - partial, 186–189
 - signed, 241–242
 - vexillary, 222
- Pfaffian, 233, 238, 392–393
 - formula for degeneracy locus, 253–254
 - Laplace expansion, 393, 394
- multi-linearity, 393, 402
- multi-Schur, 250–251
- Schur’s identity, 393
- piecewise polynomial function, 122–123

442

Subject Index

- in equivariant Chow
 - cohomology, 124
- Pieri rule, 60, 138–140
- Plücker embedding, 45
- Poincaré dual basis, 37, 38, 373
 - for G/B , 274
 - for G/P , 278
 - for Bott–Samelson variety, 333–336
 - for flag variety, 161–163
 - for Grassmannian, 55–56, 135–137
 - for isotropic flag variety, 248
 - from transverse subvarieties, 58
- Poincaré duality, 37–39, 372
 - in Schubert basis, 135–137, 162, 248, 274, 278
 - toric varieties, 122
- Poincaré isomorphism, 372
- polytope, 114
 - f -numbers, 122
 - h -numbers, 122
 - simple, 115
- positivity
 - in cohomology of Grassmannian, 145
 - in Schubert calculus, 6, 60, 169, 340, 356–369
 - in Schur expansion, 44, 60
 - of polynomial in Chern classes, 43
- principal bundle, 9–12
 - associated to vector bundle, 10
 - locally trivial, 11, 288
 - universal, 12, 416
- projection formula, 35, 309, 339
- projective bundle, 10
 - cohomology of, 4, 23
- projective line
 - as T -curve, 98
 - cohomology of, 36
 - localization, 100
- projective space
 - as homogeneous space, 33
 - cohomology of, 33, 47–48
 - equivariant cohomology of, 23–24
 - fixed points, 63
 - localization, 63, 101–102
 - T -curves, 98
- proper intersection, 357, 358, 375
- pullback, 28
- Q -polynomial, 232–233, 393–404
 - and cohomology of Lagrangian Grassmannian, 234
- basis for Γ , 396
- double, 250–252, 262, 398–404
- interpolation property, 251, 400–403
- Pfaffian formula, 394
- raising operator formula, 395
- represents Schubert class, 252, 403
- tableau formula, 397–398

- quaternionic Grassmannian, 283
- Quillen–Suslin theorem, 120
- raising operator, 395–396
- rank condition, 7, 153, 185, 193–194, 199
- rank function, 152, 185–189, 191, 195
- reduced expression, 160, 268
- reduced word, 268, 329, 330, 339
- reductive group, 63, 266
- reflection, 267
 simple, 268
- representation, 17
 dual, 203
 of GL_n , 43
 of torus, 18
 standard, 17
- restriction, *see* localization
- Richardson variety, 163–164, 179
- root, 266
- root lattice, 266–268, 298
- root subgroup, 267
- Schubert basis, 59
 for G/B , 272–273
 for G/P , 277
 for flag variety, 154, 172
 for Grassmannian, 128
 for isotropic flag variety, 247
 for projective space, 59
- Schubert calculus, 6, 126–151
- equivariant, 59, 137–141, 145–149
- Schubert cell
 in G/B , 271
 in G/P , 276
 in flag variety, 153–156
 in Grassmannian, 126–128
 in isotropic flag variety, 245–247
 in partial flag variety, 171
 opposite, 135, 162, 246
- Schubert class, 6
 interpolation property, 132, 250, 261
 restriction to fixed point, 131–132, 156, 261, 296–299, 339–343
 stable, 174–175, 210–211, 214–216, 248–249
- Schubert divisor, 342, 353
 restriction to fixed point, 169, 298
- Schubert polynomial, 6–8, 164–169, 179, 254–262
 back-stable, 222
 basis for polynomial ring, 176–177, 218, 259
 characterized by
 interpolation, 178, 222–223, 260–261
 duality property, 192, 199
 for infinite flag variety, 216, 263
 for isotropic flag variety, 254–259

- represents degeneracy locus,
 - 8, 191, 195, 219–221
- represents Schubert class, 7,
 - 167, 172, 175, 256
- stability, 173–176, 218, 221,
 - 255, 262
- structure constants, 177,
 - 205–206
- universal property, 199–201
- Schubert variety
 - as degeneracy locus, 183, 191
 - Cohen–Macaulay, 180
 - in G/B , 271–272
 - in G/P , 276
 - in flag bundle, 154
 - in flag variety, 153–156
 - in Grassmannian, 128
 - in infinite flag variety,
 - 214–219
 - in isotropic flag variety,
 - 245–247
 - in Lagrangian
 - Grassmannian, 249
 - in partial flag variety, 171
 - is closure of Schubert cell,
 - 160
 - matrix, 206
 - nonsingular in codimension
 - one, 347, 352
 - nonsingularity criterion,
 - 344–348
 - opposite, 135, 162, 246
- Schur Q -function, 232, 238,
 - 254, 394–398
- Schur determinant, 42, 183, 238
 - basis for cohomology of flag
 - variety, 54
 - basis for cohomology of
 - Grassmannian, 51
 - basis for polynomial ring, 52
 - multi-, 129, 143
- Schur function, 40–44
 - as sum over tableaux, 41
 - basis for symmetric
 - functions, 42, 53
 - double, 132–135, 150
 - represents Schubert class,
 - 135
- Schur module, 43
- Segre class, 315–318
- Segre variety, 184
- self-intersection formula, 35,
 - 68, 131, 321
- semisimple group, 265
 - adjoint, 268
 - simply connected, 268
- shelling, 115
- signed permutation, 240–242
 - Grassmannian, 242, 262
- simple transposition, 160
- simply connected group, 268
- slice theorem, 63–64, 77, 100
- smooth morphism, 375
- special group, 24, 288
- special linear group, 20, 65
- specialization homomorphism,
 - 388–391
- spherical variety, 91
- split Grassmannian, 20
- splitting principle, 21, 380
- stack, 8

- standard flag, 153, 229
- Stanley–Reisner ring, 117–122, 124
- Steiner’s problem, 79, 90
- Stiefel variety, 9
- structure constant, 6
 for G/P , 350–369
 for flag variety, 169–172
 for Grassmannian, 137–141
 for projective space, 60, 61
 inductive characterization, 140–141, 150, 170, 353–356
- submersion, 375
- subword, 161
- Sumihiro’s theorem, 94, 110
- support
 of cohomology class, 195
- Sylvester identity, 71, 77
- symmetric function
 complete homogeneous, 41, 134
 elementary, 41
 ring of, 42, 213
 Stanley, 223, 264
 super-, 212–213, 218
- symmetric group, 269
- symmetric map, 226, 234
- symplectic form, 228
- symplectic group, 229
 cohomology of, 22, 282–283
- T -curve, 98
- tangent cone, 319
 to Schubert variety, 349
- tangent weight
 relatively prime, 105
 to G/B , 273
 to G/P , 275
 to Bott–Samelson variety, 328
 to flag variety, 155
 to Grassmannian, 131
 to invariant curve, 295
 to isotropic flag variety, 245
 to Lagrangian
 Grassmannian, 243
 to Schubert variety, 131, 156, 246, 273, 277
- Thom class, 379
- Thom isomorphism, 379
- toric variety, 113–124, 331
 cohomology of, 115–117
 invariant divisors, 116
- torus, 26–27
 cohomology of, 2
 maximal, 265
- transversality
 for flag bundles, 359–362
 Kleiman–Bertini, 6, 357–362
 of opposite Schubert
 varieties, 137, 246
 Poincaré dual bases, 58
- trivial action, 32
- tubular neighborhood, 64
- unipotent group, 31, 266
- unipotent radical, 31, 266, 274
- universal coefficient theorem, 371
- universal principal bundle, 12
- Veronese surface, 80

446

Subject Index

- in space of conics, 83
- tangent weights, 85
- weight diagram, 85
- weight lattice, 266–268
- Weyl group, 268–270, 275
 - action on cohomology of G/B , 296, 301–302
 - dot and star actions, 309
 - invariants, 284–288
 - left- and right-handed
 - actions, 303, 306, 309
 - right action on G/B ,
 - 281–282, 301–302, 337
- Whitney formula, 17, 380
 - for relations in cohomology, 47, 49, 234, 235
- Yamanouchi word, 145
- Young diagram, 40
 - shifted, 240, 249, 397
- Young subgroup, 171
- Young tableau
 - reverse barred ν -bounded, 146
 - semistandard, 40–41, 44, 133, 145, 158
 - shifted primed, 397–398