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Topological Duality for Distributive Lattices

Introducing Stone–Priestley duality theory and its applications to logic and theoretical computer science, this book equips graduate students and researchers with the theoretical background necessary for reading and understanding current research in the area.

After giving a thorough introduction to the algebraic, topological, logical, and categorical aspects of the theory, the book covers two advanced applications in computer science, namely in domain theory and automata theory. These topics are at the forefront of active research seeking to unify semantic methods with more algorithmic topics in finite model theory. Frequent exercises punctuate the text, with hints and references provided.

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Topological Duality for Distributive Lattices

Theory and Applications

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Preface

This book is a course on Stone–Priestley duality theory, with applications to logic and the foundations of computer science. Our target audience includes both graduate students and researchers in mathematics and computer science. The main aim of the book is to equip the reader with the theoretical background necessary for reading and understanding current research in duality and its applications. We aim to be didactic rather than exhaustive; however, we do give technical details whenever they are necessary for understanding what the field is about.

Distributive lattice structures are fundamental to logic, and thus appear throughout mathematics and computer science. The reason for this is that the notion of a distributive lattice is extremely basic: It captures a language containing as its only primitives the logical operators "or," "and," "true," and "false." Distributive lattices are to the study of logic what rings and vector spaces are to the study of classical algebra. Moreover, distributive lattices also appear in, for example, ring theory and functional analysis.

The mathematical kernel that makes duality theory tick is the fact that the structure of a lattice can be viewed in three equivalent ways. A distributive lattice is all of the following:

(a) a partially ordered set satisfying certain properties regarding upper and lower bounds of finite sets;
(b) an algebraic structure with two idempotent monoid operations that interact well with each other; and
(c) a basis of open sets for a particular kind of topological or order-topological space.

The first part of the book will define precisely the vague notions in this list ("certain properties," "interact well," "a particular kind of"), and will prove that these are indeed three equivalent ways of looking at distributive lattices. The correspondence between algebraic and topological structure in the last two items of the list can be cast in a precise categorical form, and is then called a *dual equivalence* or simply *duality*. This duality identifies an exciting, almost magical, and often highly useful intersection point of algebra and topology.

Historically, Stone showed in the 1930s that distributive lattices are in a duality with spectral spaces: a certain class of topological spaces with a non-trivial specialization order, which are also the Zariski spectra of rings. Stone's duality for distributive lattices is especially well known in the more restrictive setting of Boolean algebras, obtained by adding an operator "not" to the lattice signature, which satisfies the usual rules of logic: de Morgan's laws and excluded middle. The restriction of Stone's duality to Boolean algebras shows that they are in a duality with compact Hausdorff zero-dimensional spaces. While the spaces associated to Boolean algebras are better known than the slightly more general ones associated to distributive lattices, the latter are vastly more versatile, having as continuous retracts, among others, *all* compact Hausdorff spaces, including connected spaces such as the unit interval of the reals.

Nevertheless, Stone's duality for distributive lattices was for at least 30 years seen by many as a lesser sibling of his duality for Boolean algebras, at least partly due to the fact that the spaces that figure in it are not Hausdorff, and the appropriate functions between the spaces are not all the continuous ones. Priestley's seminal work in the 1970s lifted this obstacle, by giving a first-class role to the specialization order that figures in Stone's spectral spaces. Priestley reframed Stone's duality as one between distributive lattices and certain *partially ordered* topological spaces, now called Priestley spaces. The first goal in this book is to build up the necessary mathematics to prove Priestley's duality theorem, which we do in Chapter 3; we also show there how it easily specializes to the case of Boolean algebras. Building up to this first main result, Chapters 1 and 2 will teach the foundations of order theory and topology that we rely on in the rest of the book.

A unique feature of this book is that, in addition to developing general duality theory for distributive lattices, we also show how it applies in a number of areas within the foundations of computer science, namely, modal and intuitionistic logics, domain theory, and automata theory. The use of duality theory in these areas brings to the forefront how much their underlying mathematical theories have in common. It also prompts us to upgrade our treatment of duality theory with various enhancements that are now commonly used in state-of-the-art research in the field. Most of these enhancements make use of *operators* on a distributive lattice: maps between lattices that only preserve part of the lattice structure.

The simplest kind of operator is a map between lattices that respects the structure of "and" and "true," but not necessarily "or" and "false." If this notion is understood as analogous to a linear mapping in linear algebra, then it is natural to also consider more general binary, ternary, and n-ary operators on lattices, which respect the structure of "and" and "true" in each coordinate, as long as the remaining coordinates are fixed. The theory of lattices with operators, and dualities for them, was developed in the second half of the twentieth century, roughly in two main chunks. First, in the 1950s, by Jónsson and Tarski, in the case of Boolean algebras, with immediate applications to relation algebra, and the same theory was used heavily a little later and very successfully for modal logic in the form of Kripke's semantics. However, until the end of the 1980s, the duality theory for distributive lattices with additional operations developed in the form of a great number of isolated case-by-case studies. Starting with the work of Goldblatt, and also of Jónsson and this book's first author, the general theory of distributive lattices with additional operations came into a mature, more usable, form by the 1990s. This theory is developed in Chapter 4, which also contains the first applications of duality theory, to free distributive lattices, quotients and subspaces, implication-type operators, Heyting algebras, and Boolean envelopes.

In the development of the first four chapters of this book, we keep the use of category theory to a minimum. In Chapter 5, we then set the results of the earlier chapters in the more abstract and general framework of category theory. This development then also allows us to show how Priestley's duality fits well in a more general framework for the interaction of topology and order, which had been developed by Nachbin shortly before. In Chapter 6, we show how the various classes of topological spaces with and without order, introduced by Stone, Priestley, and others, all relate to each other, and how they are in duality with distributive lattices and their infinitary variant, frames.

Chapters 7 and 8 contain two more modern applications of duality theory to theoretical computer science, namely to domain theory and to automata theory, respectively. The domain theory that we develop in Chapter 7 is organized around three separate results: Hoffmann–Lawson duality; the characterization of those dcpos and domains, respectively, that fall under Stone duality; and Abramsky's celebrated 1991 Domain Theory in Logical Form paper.

The duality-theoretic approach to automata theory that we develop in Chapter 8 originates in work linking profinite methods in automata theory with duality theory (Gehrke et al., 2008). It is organized around a number of related results, namely: finite syntactic monoids can be seen as dual spaces, and the ensuing effectivity of this powerful invariant for regular languages; the free profinite monoid is the dual of the Boolean algebra of regular languages expanded with residuation operations and, more generally, topological algebras on Boolean spaces are duals of certain Boolean algebras extended by residual operations. As an extended application example, we use duality to give a profinite equational characterization for the class of piecewise testable languages; and we end by discussing a characterization of those profinite monoids for which the multiplication is open.

How to Use This Book

This is a textbook on distributive lattices, spectral spaces, and Stone and Priestley dualities as they have developed and are applied in various areas at the intersection of algebra, logic, and theoretical computer science. Our aim is to get in a fairly full palette of duality tools as directly and quickly as possible, then to illustrate and further elaborate these tools within the setting of three emblematic applications: semantics of propositional logics; domain theory in logical form; and the theory of profinite monoids for the study of regular languages and automata. The text is based on lecture notes from a 50-hour course in the *Master Logique Mathématique et Fondements de l'Informatique* at Université Paris 7, which ran in the winters of 2013 and 2014. The fact that it is based on notes from a course means that it reaches its goals while staying as brief and to the point as possible. The other consequence of its origin is that, while it is fully a mathematics course, the applications we aim at are in theoretical computer science. The text has been expanded a bit beyond what was actually said in the course, reaching research monograph level by the very end of the last two, Chapters 7 and 8. Nevertheless, we have focused on keeping the spirit of a lean and lively textbook throughout, including only what we need for the applications, and often deferring more advanced general theory to the application chapter where it becomes useful and relevant.

While the original course on which the book is based covered the majority of all the chapters of the book, there are several other options for its use. In particular, a basic undergraduate course on lattices and duality could treat just Chapters 1 through 3 and possibly selected parts of 4, 5, and/or 6. The applications in the second part are fairly independent and can be included as wanted, although the domain theory material in Chapter 7 requires at least skeletal versions of Chapter 5, and Chapter 6 in its entirety.

The first part, Chapters 1 through 6, is a graduate-level "crash course" in duality theory as it is practiced now. Chapter 1 introduces orders and lattices, and in particular the distributive lattices that we will be concentrating on, as well as the finite case of Stone duality, where topology is not yet needed. Chapter 2 introduces the topological side of the dualities. In this chapter, we elaborate the interaction between order and topology, which is so central to the study of spaces coming from algebraic structures. For this purpose we have bent our philosophy of minimum content somewhat by introducing the class of stably compact spaces and Nachbin's equivalent class of compact ordered spaces. We believe that this setting provides the right level of generality for understanding the connection between Stone's original duality for spectral spaces and Priestley duality. The class of stably compact spaces, being the closure of spectral spaces under continuous retracts, is also a more robust setting than spectral spaces for a number of further applications that we do not cover in this book, such as continuous domain theory and duality for sheaf representations of algebras. The basic mathematical content of Priestley duality is given in Chapter 3. Chapter 4 introduces the most important general methods of modern duality theory: duality for additional operations and sub-quotient duality, which then allows us to immediately give first applications to propositional logics. Chapter 5 then introduces categorical concepts such as adjunctions, dualities, filtered colimits, and cofiltered limits, which play a fundamental role in duality theory. This allows us to give a full categorical account of Priestley duality by the end of the chapter. Chapter 6 treats the Omega-Point duality and Stone's original duality for distributive lattices and makes the relationship between these dualities and Priestley's version clear.

The duality theory developed in the first six chapters of the book is applied to two different parts of theoretical computer science in the last two chapters, which provide an entry into research-level material on these topics. These two chapters are independent from each other, and have indeed traditionally been somewhat separate in the literature, but our treatment here shows how both topics in fact can be understood using the same duality-theoretic techniques that we develop in the first part of the book. When using this book for a course, a lecturer can freely choose material from either or both of these chapters, according to interest. Chapter 7, on domain theory, contains a duality-theoretic exposition of the solutions to domain equations, a classical result in the semantics of programming languages. Chapter 8 develops a duality theory for algebraic automata theory, and shows in particular how finite and profinite monoids can be viewed as instances of the dual spaces of lattices with operators that we study in this book.

We have given some bibliographic references throughout the text. We want to emphasize here that these references are not in any way meant to give an exhaustive bibliography for the vast amount of existing research in duality theory. They are rather intended as useful entry points into the research literature appropriate for someone learning this material, who will then find many further references there. Furthermore, at the end of several chapters, we have added a small number of additional notes giving technical pointers pertaining to specific topics discussed there – again, we do not mean to imply exhaustivity. When we introduce special or less standard notation, we use "Notation" blocks, which are occasionally numbered when we need to refer back to them later. The book ends with a listing of the most-used notations and an index of concepts.

In each chapter, all numbered items follow one and the same counter, with the exception of exercises, which follow a separate numbering, indicating not only chapter but also section. On the topic of exercises: this book contains many of them, varying greatly in difficulty. In earlier chapters, many of the exercises are routine verifications, but necessary practice for a learner who wants to master the material.

Especially in later chapters, there are exercises that could be viewed as small research projects, although we refrain from stating open problems as exercises: for the less obvious exercises, we have included hints and references where available.

Comparison to Existing Literature and Innovative Aspects

The first part of this book, Chapters 1 through 6, covers quite classical material and may be compared to existing textbooks. The closest are probably *Distributive Lattices* (Balbes and Dwinger, 1975) and *Introduction to Lattices and Order* (Davey and Priestley, 2002). Another classical gentle introduction to the field, but focusing more on point-free topologies and frame theory than we do here, is *Topology via Logic* (Vickers, 1989). Of these, Balbes and Dwinger (1975) is probably the closest in spirit to our treatment, as it gets to the duality quickly and then applies it. However, that book's applications to algebras of propositional logic focus on varieties that are less central today. Davey and Priestley's textbook has been very successful and has in particular managed to attract a theoretical computer science readership to these topics. However, it focuses more on the lattices and order per se and the duality is covered only as one of the final crowning chapters. Davey and Priestley's book is therefore an excellent way in to ours, and we recommend it as supplemental reading in case students need additional details or to build up mathematical maturity. The textbooks *Introduction to Boolean Algebras* (Givant and Halmos, 2008) and *Duality Theories for Boolean Algebras with Operators* (Givant, 2014) are also relevant but are of course restricted to the Boolean setting. There are also a number of classical references in lattice theory by Grätzer, the most recent versions being *Lattice Theory: Foundation* (Grätzer, 2011) and *General Lattice Theory* (Grätzer, 2003), both of which contain material on duality theory and its applications to lattice theory.

Here we aim to get the dualities in place as soon as possible and then use them. Where we differ the most from the existing books within this first part is with our emphasis on the interaction between order and topology in Chapter 2, and in placing Priestley duality within the wider context of category theory (Chapter 5) and Omega-point duality (Chapter 6). Chapter 2 provides a textbook-level didactic account of the interaction between topology and order culminating with the equivalence between Nachbin's compact-ordered spaces and stably compact spaces. In Chapter 4 we develop duality theory methods for analyzing the structure of distributive lattices and operators on them. All of these topics have become central in research in recent decades but are so far difficult to access without delving in to the specialized literature.

Perhaps the most important omission of this book is the theory of canonical extensions, which was central to the previously mentioned work of Jónsson and others, in addition to duality. While this theory is very close to the hearts of both authors of this book, and closely related to duality theory, this book is not about that, and canonical extensions thus do not play a big role in this book, at least not explicitly. Still, we will occasionally make reference to canonical extensions where appropriate. Along with and closely related to this omission, we decided to take a point-set rather than a point-free approach to the topics of this book. Point-free approaches focus on the algebraic side of duality and thus avoid the point-set world of topology, which inherently involves non-constructive principles. Duality is in a sense the justification for the point-free approach since it makes the link between the algebraic and the point-set worlds. In this book we remain fully anchored in the set-theoretic approach to topology, in particular making use of the axiom of choice as necessary. We do this as it is more easily accessible for a general audience, and because our end applications in denotational semantics and profinite algebras in automata theory are, in their currently practiced form, focused on point-set topology. That being said, our focus on duality shows the way and familiarizes the reader with the dual, point-free approach, thus making them ready to embrace this approach. In this direction, one of our hopes with this book is that it will entice some readers to learn about canonical extensions and related point-free techniques. We believe the technique of canonical extensions to be complementary to, and at least as important as, duality, but so far less well established in the literature.

Many research monographs include similar material to that of the first part of this book, but are not explicitly targeted at readers who are first learning about the field, while this is a primary aim of our book. Classical such monographs, closest in content to the first part of this book, are *Stone Spaces* (Johnstone, 1986) and *A Compendium of Continuous Lattices* (Gierz et al., 1980), re-edited as *Continuous Lattices and Domains* (Gierz et al., 2003). More recently, the monograph *Spectral Spaces* (Dickmann et al., 2019) studies the same class of spaces as we do in this book, but comes from a ring-theoretic perspective and emphasizes less the order-theoretic aspects. The monograph *Non-Hausdorff Topology and Domain Theory* (Goubault-Larrecq, 2013) is close in spirit to our treatment in Chapter 2, especially in its treatment of stably compact spaces, and also addresses a theoretical computer science audience. A difference with our treatment here is that Goubault-Larrecq (2013) is focused on non-Hausdorff topologies and therefore does not treat the (Hausdorff) patch topology as central, as we do here. Related to our Chapter 6 is the monograph *Frames and Locales* (Picado and Pultr, 2012) focused on frames and point-free topology, and Chapter 6 of this book can be used as a preparation for jumping into that work.

The applications to domain theory and automata theory are treated in Chapters 7 and 8, respectively. These two applications, and in particular the fact that we treat them in one place, as applications of a common theory, are perhaps the most innovative and special aspects of this book. Domain theory is the most celebrated application of duality in theoretical computer science and our treatment is entirely new. Automata theory is a relatively new application area for duality theory and has never been presented in textbook format before. More importantly, both topics are at the forefront of active research seeking to unify semantic methods with more algorithmic topics in finite model theory. While previous treatments remained focused on the point of view of domains/profinite algebra, with duality theory staying peripheral, a shared innovative aspect of the presentations of these topics in this book is that both are presented squarely as applications of duality.

Finally, a completely original contribution of this book, which emerged during its writing, precisely thanks to our treatment of the two topics as an application of a common theory, is the fact that a notion we call "*preserving joins at primes*" turns out to be central in both the chapter on domain theory and in that on automata theory. This notion was introduced in the context of automata theory and topological algebra in Gehrke (2016); its application to domain theory is new to this book and reflects a key insight of Abramsky's Domain Theory in Logical Form. We believe this point to be an exciting new direction for future research in the field that we hope some readers of this book will be inspired to take up.

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