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Mai Gehrke, Sam van Gool
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Order and Lattices

In this chapter, we first introduce basic notions from order theory: preorders, partial orders, and lattices. We then zoom in on distributive lattices. In the finite case, we prove from first principles a duality theorem, which is a blueprint for the more advanced duality theorems that follow later in this text.

1.1 Preorders, Partial Orders, and Suprema and Infima

A binary relation $\leq$ on a set P is called

- *reflexive* if $p \leq p$ for all $p \in P$,
- *transitive* if $p \leq q \leq r$ implies $p \leq r$ for all $p, q, r \in P$,
- *anti-symmetric* if $p \leq q$ and $q \leq p$ imply $p = q$ for all $p, q \in P$,
- a *preorder* if it is reflexive and transitive, and
- a *partial order* if it is reflexive, transitive, and anti-symmetric.

A *preordered set* is a tuple $(P, \leq)$ with $\leq$ a preorder on the set P. A *poset* (short for partially ordered set) is a pair $(P, \leq)$ with $\leq$ a partial order on the set P. Two elements p and q are *comparable* in a preorder $\leq$ if at least one of $p \leq q$ and $q \leq p$ holds, and *incomparable* otherwise. The adjective "partial" in "partial order" refers to the fact that not all elements in a partial order are comparable. A preorder is called *total* or *linear* if any two of its elements are comparable. A *total order* or *linear order* or *chain* is a total preorder which is moreover anti-symmetric. A poset is called an *anti-chain* if no distinct elements are comparable. The *strict part* of a partial order is the relation $<$ defined by $p < q$ if, and only if, $p \leq q$ and $p \neq q$. Notice that, if we have access to equality, then to specify a partial order $\leq$, it suffices to specify its strict part $<$, from which we can then define $p \leq q$ if, and only if, $p < q$ or $p = q$. An *equivalence relation* is a preorder $\leq$ which is moreover *symmetric*, that is, $p \leq q$ implies $q \leq p$ for all $p, q \in P$. In this context, comparable elements are called *equivalent*.

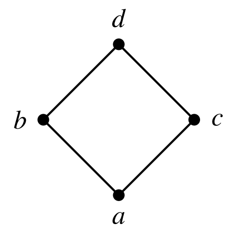

Figure 1.1 The "diamond" poset $(D, \leq)$.

Example 1.1 The *Hasse diagram* of the "diamond" poset $D = \{a, b, c, d\}$, with partial order $\leq$ whose strict part is $\{(a, b), (a, c), (a, d), (b, d), (c, d)\}$ is depicted in Figure 1.1. This partial order is not linear, because we have neither $b \leq c$ nor $c \leq b$.

Notice that, in the above example, even though $a \leq d$, we did not draw an edge between a and d in the Hasse diagram. This is due to the fact that $a \leq d$ can be inferred by transitivity from the order relations $a \leq b$ and $b \leq d$, which are depicted in the diagram. Thus, we only need to draw the "covering" relations in the diagram.

We now give the general definition of *Hasse diagram*.

Definition 1.2 For elements p and q of a poset P, we say that q *covers* p if $p < q$ and there is no $r \in P$ such that $p < r < q$. We denote this relation by $p \prec q$. The elements of a poset are represented in the Hasse diagram as nodes. An edge is drawn from a node p to a node q whenever q covers p. In addition, in order not to have to indicate the direction of edges by arrows, the convention is that moving up along an edge in the diagram corresponds to moving up in the order. Thus, in particular, points drawn at the same height are incomparable.

All finite posets are represented by their Hasse diagrams, as are some infinite ones. However, for most infinite posets, the Hasse diagram does not suffice for capturing the order. For example, the usual order on the unit interval has an empty covering relation.

There are several interesting classes of maps between preordered sets. (We use the word "map" interchangably with "function" throughout this book.) Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be preordered sets and $f : P \to Q$ a function. The function f is called

- *order preserving* or *monotone* if $p \leq_P p'$ implies $f(p) \leq_Q f(p')$ for all $p, p' \in P$,
- *order reflecting* if $f(p) \leq_Q f(p')$ implies $p \leq p'$ for all $p, p' \in P$,
- an *order embedding* if it is both order preserving and order reflecting, and
- an *order isomorphism* if it is order preserving and has an order-preserving inverse.

Note that order embeddings between posets are always injective, but not all injective order-preserving maps between posets are order embeddings! See Exercise 1.1.4(a). A function f between preordered sets is an order isomorphism if, and only if, f is a surjective order embedding, see Exercise 1.1.4(b). If $(P, \leq_P)$ is a preordered set and P' is a subset of P, then the *inherited order* on P' is the intersection of $\leq_P$ with $P' \times P'$, that is, it is such that the inclusion map $i \colon P' \hookrightarrow P$ is an order embedding.

An elementary but important operation on preorders is that of "turning upside down." If P is a preorder, we denote by P^{op} the *opposite* of P, that is, the preorder with the same underlying set as P, but with preorder $\leq'$ defined by $p \leq' q$ if, and only if, $q \leq p$, where $\leq$ denotes the original preorder on P. A function $f \colon P \to Q$ is called *order reversing* or *antitone* if it is order preserving as a function $f \colon P^{\mathrm{op}} \to Q$, that is, if for all $p, p' \in P$, if $p \leq_P p'$, then $f(p') \leq_Q f(p)$. An *anti-isomorphism* between P and Q is, by definition, an isomorphism between P^{op} and Q.

Notation 1.3 Throughout this book, when $f \colon X \to Y$ and $g \colon Y \to Z$ are functions, we write $g \circ f$ for the *functional composition* of f and g, to be read as "g after f," that is, $g \circ f$ is the function $X \to Z$ defined by $(g \circ f)(x) = g(f(x))$ for all $x \in X$. We will sometimes omit the symbol $\circ$ and just write gf. In Notation 4.37, we will introduce a slightly different notation for *relational composition*, as is common in the literature.

Example 1.4 For any natural number n, the finite set $\mathbf{n} := \{0, 1, \ldots, n-1\}$ is totally ordered by the usual ordering of natural numbers.

Example 1.5 The sets of natural numbers $\mathbb{N}$, integers $\mathbb{Z}$, rational numbers $\mathbb{Q}$, and real numbers $\mathbb{R}$, with the usual orders, are total orders.

Example 1.6 On the set of natural numbers $\mathbb{N}$, define a relation $\leq$ by

$$p \leq q \iff p = 0 \text{ or } (p \neq 0 \text{ and } q \neq 0).$$

Note that $\leq$ is a preorder, but not a partial order. We define the *poset reflection* of this preorder as follows (see Exercise 1.1.5 for the general idea). Consider the quotient of $\mathbb{N}$ by the equivalence relation that identifies all non-zero numbers; denote this quotient by P, and equip it with the least preorder such that the quotient map $\mathbb{N} \to P$ is order preserving. Then, P is a poset, and any other order-preserving function from $\mathbb{N}$ to a poset $(Q, \leq)$ factors through it.

Example 1.7 Let F be a set of formulas in some logic with a relation of derivability $\vdash$ between formulas of F. More concretely, F can be the set of sentences in a first-order signature and $\vdash$ derivability with respect to some first-order theory. The relation $\vdash$ is rarely a partial order, as there are usually many syntactically different formulas which are mutually derivable in the logic. The poset reflection (see Exercise 1.1.5) consists of the $\vdash$-equivalence classes of formulas in F.

Example 1.8 Denote by $\mathbf{2}^*$ the set of finite sequences over the two-element set $\mathbf{2} = \{0, 1\}$. The binary operation of *concatenation* is defined by juxtaposition of sequences. That is, given sequences $p, r \in \mathbf{2}^*$ of length n and m, respectively, pr is the sequence of length $n + m$ whose i^{th} entry is the i^{th} entry of p if $i \leq n$ and is the $(i - n)^{\text{th}}$ entry of r otherwise.

(a) For $p, q \in \mathbf{2}^*$, define

$$p \leq_P q \iff \text{there exists } r \in \mathbf{2}^* \text{ such that } pr = q.$$

Note that $\leq_P$ is a partial order on $\mathbf{2}^*$ (see Exercise 1.1.2). The poset $(\mathbf{2}^*, \leq_P)$ is called the *full infinite binary tree*. The partial order $\leq_P$ on $\mathbf{2}^*$ is called the *prefix order*.

(b) For $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_m) \in \mathbf{2}^*$, define $p \leq_{\text{lex}} q$ if, and only if, p is a prefix of q or, at the least index such that $p_i \neq q_i$, we have $p_i \leq q_i$ in $\mathbf{2}$. Note that $\leq_{\text{lex}}$ is a total order on $\mathbf{2}^*$ (see Exercise 1.1.3). The partial order $\leq_{\text{lex}}$ is called the *lexicographic* or *dictionary order* on $\mathbf{2}^*$.

We define the fundamental notions of supremum and infimum.

Definition 1.9 Let $(P, \preceq)$ be a preorder. Let $S \subseteq P$.

- An element s_0 of P is called a *lower bound* of S if $s_0 \preceq s$ for all $s \in S$.
- An element s_1 of P is called an *upper bound* of S if $s \preceq s_1$ for all $s \in S$.
- A lower bound s_0 of S is called an *infimum* or *greatest lower bound* of S if, for any lower bound s' of S, $s' \preceq s_0$.
- An upper bound s_1 of S is called a *supremum* or *least upper bound* of S if, for any upper bound s' of S, $s_1 \preceq s'$.

In the special case where $S = \emptyset$, an element s_0 which is a supremum of S is called a *bottom* or *minimum* element of P, meaning that $s_0 \preceq s$ for all $s \in S$. Similarly, an element s_1 which is an infimum of S is called a *top* or *maximum* element of P.

In a poset, any set has at most one infimum and at most one supremum (see Exercise 1.1.6). If a unique infimum of a subset S exists, it is denoted by $\bigwedge S$ and is also known as the *meet* of S. The supremum of S, if it exists uniquely, is denoted by $\bigvee S$ and is known as the *join*. In the case where $S = \{a, b\}$, we also write $a \wedge b$ and $a \vee b$, and if $S = \{a_1, \dots, a_n\}$ we write $a_1 \vee \cdots \vee a_n$ and $a_1 \wedge \cdots \wedge a_n$. The bottom element, if it exists, is denoted by $\bot$ or 0, and the top element by $\top$ or 1. If S is a subset of a poset P, we denote the set of *maximal* elements in S by $\max(S)$; that is,

$$\max(S) := \{s \in S \mid \text{ for all } s' \in P, \text{ if } s \leq s' \text{ and } s' \in S, \text{ then } s' = s\}.$$

Similarly, the set of minimal elements in S is denoted by $\min(S)$. In contrast to maximal elements of a set, the supremum of a set does not need to belong to the set itself. Note that non-empty subsets of a poset may not have any minimal or maximal

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elements; see the examples below. Also, postulating the existence of maximal or minimal elements in certain posets is related to choice principles; see our discussion of *Zorn's lemma*, Lemma 2.7, in Chapter 2.

Remark 1.10 There are subtle but important differences between the words "maximum," "maximal," and "supremum." An element is *maximal* in a subset S of a poset if there is no other element in S that lies strictly above it, while it is a *maximum* element in S if all other elements of S lie below it. Note that in a *totally* ordered set, the concepts maximal and maximum are equivalent, but not in general. Finally, an important distinction between these two concepts and that of supremum is that, for an element to be a supremum, it is *not* a requirement that it lies in the set itself, while this is part of the definition for maximal and maximum elements, see Exercise 1.1.7. For this reason, the supremum of a set S depends on the ambient poset, see Exercise 1.2.6.

Infima and suprema may fail to exist. There are three different situations in which this can happen: a set can either have no lower (or upper) bounds at all; or its set of lower (or upper) bounds has incomparable maximal (or minimal) elements; or for infinite sets, the set of upper bounds may be non-empty but not have all such above a minimal upper bound (or a non-empty set of lower bounds not all witnessed by maximal lower bounds).

We illustrate the above ideas with three examples.

Example 1.11 In the poset $(P, \leq)$ whose Hasse diagram is depicted in Figure 1.2, the set $S = \{a, b\}$ does not have an infimum, because c and d are incomparable maxim*al* lower bounds of S, and hence neither is a maxim*um* lower bound.

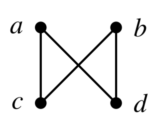

Figure 1.2 The "butterfly" poset $(P, \leq)$.

Example 1.12 In the set $\mathbb{N}$ of natural numbers with its usual total order, any subset has an infimum, which is in fact a minimum, but the only subsets having a supremum are the finite subsets. For any finite subset, the supremum is in fact a maximum.

Example 1.13 In the set $\mathbb{Q}$ of rational numbers with its usual total order, the subset $\{\frac{1}{n} \mid n \in \mathbb{N}_{\geq 1}\}$ has an infimum, 0, but it does not have a minimum. Furthermore $\{q \in \mathbb{Q} \mid q \leq \sqrt{2}\}$ has upper bounds but no least upper bound.

We end this section by introducing the important concepts of *adjunction* and *Galois connection* between preordered sets.

Definition 1.14 Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be preordered sets, and let $f : P \to Q$ and $g : Q \to P$ be functions. The pair (f, g) is called an *adjunction*, with f the *left* or *lower adjoint* and g the *right* or *upper adjoint*, provided that for every $p \in P$ and $q \in Q$,

$$f(p) \leq_Q q \text{ if, and only if, } p \leq_P g(q).$$

An adjunction between P^{op} and Q is called a *Galois connection* or *contravariant adjunction*.

The notion of adjunction between (pre)orders is very important and will come into play in many places in this book. It is also a useful precursor to the concept of adjunction between categories that we will encounter later in Definition 5.15. Exercises 1.1.8 and 1.2.14 collect some important basic facts about adjunctions between preorders, which will be used throughout the book.

Example 1.15 Let $f : \mathbb{Z} \hookrightarrow \mathbb{Q}$ be the order embedding which sends each integer $x \in \mathbb{Z}$ to itself, regarded as a rational number. The map f has a right adjoint g, which sends each rational $y \in \mathbb{Q}$ to its *floor*, that is, $g(y)$ is the largest integer below y. The map f also has a left adjoint, which sends a rational $y \in \mathbb{Q}$ to its *ceiling*, that is, the smallest integer above y.

Contravariant adjunctions occur particularly often in mathematics since, as we will see in the following example, they arise naturally any time we have a binary relation between two sets.

Example 1.16 Fix a relation $R \subseteq X \times Y$ between two sets. For any $a \subseteq X$ and $b \subseteq Y$, define the sets $u(a) \subseteq Y$ and $\ell(b) \subseteq X$ by

$$u(a) := \{y \in Y \mid \text{for all } x \in a, xRy\},$$
$$\ell(b) := \{x \in X \mid \text{for all } y \in b, xRy\}.$$

The pair of functions $u : \mathcal{P}(X) \leftrightarrows \mathcal{P}(Y) : \ell$ is a Galois connection between the posets $(\mathcal{P}(X), \subseteq)$ and $(\mathcal{P}(Y), \subseteq)$, that is, for any $a \subseteq X$ and $b \subseteq Y$, we have $b \subseteq u(a)$ if, and only if, $a \subseteq \ell(b)$.

The name Galois connection refers to the work of Galois in the theory of rings and fields in which the central object of study is a Galois connection induced by the binary relation between elements and automorphisms of a field given by the element being fixed by the automorphism. While this example is historically important, it is not so central to the topics of this book. So if you are not familiar with Galois theory, we provide the following two classical examples, in order theory and logic, respectively. First, in the special case when R is a preorder on a set X, $u(a)$ is the set of *common upper bounds* for the elements of a, and $\ell(b)$ is the set of *common* *lower bounds* for the elements of b. Second, the Galois connection between theories and model classes studied in logic is also a special case of the Galois connection (u, ℓ), as follows. Suppose that S is a set of structures, F is a set of logical formulas, and suppose we are given a relation of "interpretation," $\models$ from S to F, where, for $M \in S$ and $\varphi \in F$, the relation $M \models \varphi$ is read as "φ holds in M." Then, in the Galois connection of Example 1.16, u sends a set of models a to its *theory*, that is, the set of formulas that hold in every model of a, and ℓ sends a set of formulas b to its *class of models*, that is, the collection of models in which every formula from b holds.[1] Finally, not only are Galois connections obtained from a binary relation as in Example 1.16 omnipresent in mathematics, but, as we will see later, by topological duality theory, all Galois connections between distributive lattices are of this form; see in particular Proposition 5.39 and Exercise 5.4.1.

Exercises for Section 1.1

Exercise 1.1.1 Sketch the Hasse diagrams for the preorders described in Examples 1.4, 1.6, and 1.8(a).

Exercise 1.1.2 For any set A, let A^* denote the set of finite sequences of elements of A. Prove that the relation $\leq_P$ on A^* defined by

$$u \leq_P v \overset{\text{def}}{\iff} \text{there exists } w \in A^* \text{ such that } uw = v$$

is a partial order.

Note. This is a special case of the opposite of the so-called Green preorder $\leq_{\mathcal{R}}$, which exists on any monoid.

Exercise 1.1.3 Consider the relation $\leq_{\text{lex}}$ on $\mathbf{2}^*$ defined in Example 1.8(b).

(a) Prove that $\leq_{\text{lex}}$ is a total order.
(b) Prove that, even though $\mathbf{2}^*$ is infinite, the total order $\leq_{\text{lex}}$ is the transitive closure of its covering relation. That is, show that $p \leq_{\text{lex}} q$ if, and only if, there are $r_0, \dots, r_n$ with $p \prec r_0 \prec \dots r_n \prec q$.

Exercise 1.1.4 (a) Give an example of an injective order-preserving map between posets which is not an order embedding.
(b) Prove that a surjective order embedding between posets is an order isomorphism.

Exercise 1.1.5 If $(P, \leq)$ is a preordered set, define

$$p \equiv q \iff p \leq q \text{ and } q \leq p.$$

(a) Prove that $\equiv$ is an equivalence relation on P.

[1] Often, in logic, S is not a set but a proper class of models. In this case, the Galois connection is between subclasses of the class of models and subsets of the set of formulas.

(b) Prove that there is a well-defined *smallest* partial order $\leq$ on the quotient set $P/\equiv$ such that the quotient map $f \colon P \to P/\equiv$ is order preserving.
(c) Prove that, for any order-preserving $g \colon P \to Q$ with Q partially ordered, there exists a unique order-preserving $\overline{g} \colon P/\equiv \to Q$ such that $\overline{g} \circ f = g$.

The partial order $P/\equiv$ defined in this exercise is called the *poset reflection* of the preorder P.

Exercise 1.1.6 Let $(P, \preceq)$ be a preorder and $S \subseteq P$.

(a) Prove that if s_0 and s_0' are both infima of S, then $s_0 \preceq s_0'$ and $s_0' \preceq s_0$.
(b) Conclude that in a partial order, any set has at most one supremum and at most one infimum.

Exercise 1.1.7 Draw a graph with three nodes labeled “maximum,” “maximal,” and “supremum,” and directed edges denoting that the existence of one implies the existence of the other. Do any more implications hold in finite posets? In totally ordered sets? In finite totally ordered sets?

Exercise 1.1.8 Let $(P, \preceq_P)$ and $(Q, \preceq_Q)$ be preordered sets and $f : P \leftrightarrows Q : g$ a pair of maps between them.

(a) Prove that (f, g) is an adjunction if, and only if, f and g are order preserving for every $p \in P$, $p \preceq_P gf(p)$, and for every $q \in Q$, $fg(q) \preceq_Q q$.

For the rest of this exercise, assume that (f, g) is an adjunction.

(b) Prove that $fgf(p) \equiv f(p)$ and $gfg(q) \equiv g(q)$ for every $p \in P$ and $q \in Q$.
(c) Conclude that, in particular, if P and Q are posets, then $fgf = f$ and $gfg = g$.
(d) Prove that, if P is a poset, then for any $p \in P$, $gf(p)$ is the minimum element above p that lies in the image of g.
(e) Formulate and prove a similar statement to the previous item about $fg(q)$, for $q \in Q$.
(f) Prove that, if P and Q are posets, then f is the unique lower adjoint of g and, symmetrically, g is the unique upper adjoint of f.
(g) Prove that, for any subset $S \subseteq P$, if the supremum of S exists, then $f(\bigvee S)$ is the supremum of the direct image $f[S]$.
(h) Prove that, for any subset $T \subseteq Q$, if the infimum of T exists, then $g(\bigwedge T)$ is the infimum of $g[T]$.

In words, the last two items say that *lower adjoints preserve existing suprema* and *upper adjoints preserve existing infima*. In Exercise 1.2.14 of the next section we will see that a converse to this statement holds in the context of complete lattices.

Exercise 1.1.9 Let $f : P \leftrightarrows Q : g$ be an adjunction between posets. Show that f is injective if, and only if, g is surjective, and that f is surjective if, and only if, g is injective. *Hint.* Part (c) of Exercise 1.1.8 can be useful here.

1.2 Lattices

A *(bounded) lattice* is a partially ordered set L in which every finite subset has a supremum and an infimum. In fact, to be a lattice, it is sufficient that the empty set and all two-element sets have suprema and infima (see Exercise 1.2.1). A *complete lattice* C is a partially ordered set in which every subset has a supremum and an infimum. In fact, for a partially ordered set to be a complete lattice, it is sufficient that every subset has a supremum (see Exercise 1.2.2).

An interesting equivalent definition of lattices is the following. A *lattice* is a tuple $(L, \vee, \wedge, \bot, \top)$, where $\vee$ and $\wedge$ are binary operations on L (i.e., functions $L \times L \to L$), and $\bot$ and $\top$ are elements of L such that the following axioms hold:

(a) the operations $\vee$ and $\wedge$ are *commutative*, that is, $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$ for all $a, b \in L$;
(b) the operations $\vee$ and $\wedge$ are *associative*, that is, $(a \vee b) \vee c = a \vee (b \vee c)$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ for all $a, b, c \in L$;
(c) the operations $\vee$ and $\wedge$ are *idempotent*, that is, $a \vee a = a$ and $a \wedge a = a$ for all $a \in L$;
(d) the *absorption laws* $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$ hold for all $a, b \in L$; and
(e) the element $\bot$ is *neutral for* $\vee$ and the element $\top$ is *neutral for* $\wedge$, that is, $\bot \vee a = a$ and $\top \wedge a = \top$ for all $a \in L$.

Given a lattice $(L, \vee, \wedge, \bot, \top)$ according to this algebraic definition, define

$$a \leq_L b \iff a \wedge b = a. \tag{1.1}$$

Then the relation $\leq_L$ is a partial order on the set L which makes L into a lattice according to the order-theoretic definition, and the binary infimum and supremum are given by $\wedge$ and $\vee$, respectively. Conversely, given a lattice $(L, \leq)$ according to the order-theoretic definition, it is easy to check that the operations of binary join ($\vee$), binary meet ($\wedge$), and the elements $\top$ and $\bot$ make L into a lattice according to the algebraic definition, and that $\leq$ is given by (1.1), which is then also equivalent to $a \vee b = b$. The somewhat tedious but instructive Exercise 1.2.4 asks you to verify the claims made in this paragraph.

A *semilattice* is a structure $(L, \cdot, 1)$ where $\cdot$ is a commutative, associative, and idempotent binary operation, and 1 is a neutral element for the operation $\cdot$. The operation $\cdot$ can then either be seen as the operation $\wedge$ for the partial order defined by $a \leq b$ if, and only if, $a \cdot b = a$, or as the operation $\vee$ for the opposite partial order defined by $a \leq b$ if, and only if, $a \cdot b = b$. When $(L, \vee, \wedge, \bot, \top)$ is a lattice, we call $(L, \vee, \bot)$ and $(L, \wedge, \top)$ the *join-semilattice* and *meet-semilattice reducts* of L, respectively.

Remark 1.17 Many authors use the word "lattice" for posets with all binary infima and suprema, and then "bounded lattices" are those that also have $\top$ and $\bot$.

We require that all finite infima and suprema exist. Note that the non-empty finite infima and suprema are guaranteed to exist as soon as binary ones do (Exercise 1.2.1), while the empty infimum and supremum are just the $\top$ and $\bot$, respectively. In duality theory bounds are very convenient, if one does not have them, one should simply add them. Accordingly, we suppress the adjective “bounded” and use just “lattice” for the bounded ones and we will only specify when once in a while we have an unbounded lattice, sublattice, or lattice homomorphism.

Homomorphisms, Products, Sublattices, Quotients

We briefly recall a few basic algebraic notions that we will need. For more information, including detailed proofs of these statements, we refer the reader to a textbook on universal algebra, such as, for example, Burris and Sankappanavar (2000) and Wechler (1992). A function $f: L \to M$ between lattices is called a *lattice homomorphism* if it preserves all the lattice operations; that is, $f(\bot_L) = \bot_M$, $f(\top_L) = \top_M$, and $f(a \vee_L b) = f(a) \vee_M f(b)$, $f(a \wedge_L b) = f(a) \wedge_M f(b)$ for all $a, b \in L$. Lattice homomorphisms are always order preserving, and injective lattice homomorphisms are always order-embeddings (see Exercise 1.2.7). We also call an injective homomorphism between lattices a *lattice embedding*. Similarly, bijective lattice homomorphisms are always order isomorphisms, and we call a bijective homomorphism between lattices a *lattice isomorphism*.

A simple induction (Exercise 1.2.1) shows that, if a function $f: L \to M$ between lattices preserves $\bot$ and $\vee$, then it preserves all finite joins; we say that such a function *preserves finite joins*, or also that it is a *homomorphism for the join-semilattice reducts*. Note that the statement that f preserves finite joins does not in general imply anything about the preservation of any other suprema that may exist in L. As a rule, whenever we write “f preserves joins,” even though we may sometimes omit the adjective “finite,” we still only refer to preservation of finite suprema, as these are generally the only suprema that exist in a lattice. If we want to signal that f preserves other suprema than just the finite ones, then we will always take care to explicitly say so. Symmetrically, a function $f: L \to M$ that preserves $\top$ and $\wedge$ is called *finite-meet-preserving*, and the same remarks about the adjective “finite” apply here.

The *Cartesian product* of an indexed family $(L_i)_{i \in I}$ of lattices is the lattice structure on the product set $L := \prod_{i \in I} L_i$ given by pointwise operations; for example, $(\bot_L)_i = \bot_{L_i}$ and $(\top_L)_i = \top_{L_i}$ for every $i \in I$, and if $a = (a_i)_{i \in I}$, $b = (b_i)_{i \in I} \in L$, then $(a \vee b)_i = a_i \vee_{L_i} b_i$ and $(a \wedge b)_i = a_i \wedge_{L_i} b_i$. In this way, L becomes a lattice, whose partial order is also the product order, that is, $a \leq_L b \iff a_i \leq_{L_i} b_i$ for every $i \in I$, and each projection map $\pi_i : L \twoheadrightarrow L_i$ is