"This book dismantles the final, most daunting barriers to learning about moduli of higher dimensional varieties, from the point of view of the Minimal Model Program. The first chapter draws the reader in with a compelling history; a discussion of the main ideas; a visitor's trail through the subject, complete with guardrails around the most dangerous traps; and a rundown of the issues that one must overcome. The text that follows is the outcome of Kollár's monumental three-decades-long effort, with the final stones laid just in the last few years."

— Dan Abramovich, Brown University

"This is a fantastic book from János Kollár, one of the godfathers of the compact moduli theory of higher dimensional varieties. The book contains the definition of the moduli functor, the prerequisites required for the definition, and also the proof of the existence of the projective coarse moduli space. This is a stunning achievement, completing the story of 35 years of research. I expect this to become the main reference book, and also the principal place to learn about the theory for graduate students and others interested."

— Zsolt Patakfalvi, EPFL

"This excellent book provides a wealth of examples and technical details for those studying birational geometry and moduli spaces. It completely addresses several state-of-the-art topics in the field, including different stability notions, K-flatness, and subtleties in defining families of stable pairs over an arbitrary base. It will be an essential resource for both those first learning the subject and experts as it moves through history and examples before settling many of the (previously unknown) technicalities needed to define the correct moduli functor."

— Kristin DeVleming, University of Massachusetts Amherst

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Preface

The aim of this book is to generalize the moduli theory of algebraic curves – developed by Riemann, Cayley, Klein, Teichmüller, Deligne, and Mumford – to higher dimensional algebraic varieties.

Starting with the theory of algebraic surfaces worked out by Castelnuovo, Enriques, Severi, Kodaira, and ending with Mori's program, it became clear that the correct higher dimensional analog of a smooth projective curve of genus ≥ 2 is a smooth projective variety with ample canonical class. We establish a moduli theory for these objects, their limits, and generalizations.

The first attempt to write a book on higher dimensional moduli theory was the 1993 Summer School in Salt Lake City, Utah. Some notes were written, but it soon became evident that, while the general aims of a theory were clear, most of the theorems were open and even many of the basic definitions unsettled.

The project was taken up again at an AIM conference in 2004, which eventually resulted in solving the moduli-theoretic problems related to singularities; these were written up in Kollár (2013b). After 30 years, we now have a complete theory, the result of the work of numerous people.

While much of the early work focused on the construction of moduli spaces, later developments in the theory of stacks emphasized families. We also follow this approach and spend most of the time understanding families. Once this is done at the right level, the existence of moduli spaces becomes a natural consequence.

Acknowledgments

Throughout the years, I learned a lot from my teachers, colleagues, and students. My interest in moduli theory was kindled by my thesis advisor T. Matsusaka, and the early influences of S. Mori and N. I. Shepherd-Barron have been crucial to my understanding of the subject.

The original 1993 group included D. Abramovich, V. Alexeev, A. Corti, A. Grassi, B. Hassett, S. Keel, S. Kovács, T. Luo, K. Matsuki, J. McKernan, G. Megyesi, and D. Morrison; many of them have been active in this area since. My students A. Corti, S. Kovács, T. Kuwata, E. Szabó, and N. Tziolas worked on various aspects of the early theory.

I gave several lecture series about moduli. The many comments and corrections of colleagues K. Ascher, G. Farkas, M. Fulger, S. Grushevsky, J. Huh, J. Li, M. Lieblich, J. Moraga, T. Murayama, A. Okounkov, R. Pandharipande, Zs. Patakfalvi, C. Raicu, J. Waldron, J. Witaszek, and students C. Araujo, G. Di Cerbo, A. Hogadi, L. Ji, D. Kim, Y. Liu, A. Sengupta, C. Stibitz, Y.-C. Tu, D. Villalobos-Paz, C. Xu, Z. Zhuang, and R. H. Zong have been very helpful.

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Moduli theory has been developed and shaped by the works of many people. Advances in minimal model theory – especially the series of papers by C. Hacon, J. McKernan, and C. Xu – made it possible to extend the theory from surfaces to all dimensions. The projectivity of moduli spaces was gradually proved by E. Viehweg, O. Fujino, S. Kovács, and Zs. Patakfalvi. After the early works of V. Alexeev and P. Hacking, many examples have been worked out by V. Alexeev and his coauthors, A. Brunyate, P. Engel, A. Knutson, R. Pardini, and A. Thompson. Recent works of K. Ascher, D. Bejleri, K. DeVleming, S. Filipazzi, G. Inchiostro, and Y. Liu give very detailed information on important examples.

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Sections 6.5–6.6 were written with K. Altmann, while S. Kovács contributed to the writing and editing of the whole book.

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Notation

We follow the notation and conventions of Hartshorne (1977); Kollár and Mori (1998) and Kollár (2013b). Our schemes are Noetherian and separated. At the beginning of each chapter we state further assumptions. Many of the results should work over excellent base schemes, but most of the current proofs apply only in characteristic 0.

A *variety* is usually an integral scheme of finite type over a field. However, following standard usage, a *stable variety* or a *locally stable variety* is reduced, pure dimensional, but possibly reducible.

Affine n-space over a field k is denoted by $\mathbb{A}^n_k$, or by $\mathbb{A}^n(x_1, \dots, x_n)$ or $\mathbb{A}^n_{\mathbf{x}}$ if we emphasize that the coordinates are $x_1, \dots, x_n$. Same conventions for projective n-space $\mathbb{P}^n$.

The *canonical class* of X is denoted by K_X, and the *canonical sheaf* or *dualizing sheaf* by ω_X; see (1.23) for varieties and (11.2) for schemes. Since $\mathscr{O}_X(K_X) \simeq \omega_X$, we switch between the divisor and sheaf versions whenever it is convenient. Here K_X is more frequently used on normal varieties, and ω_X in more general settings. Functorial properties work better for ω_X.

A smooth proper variety X is of *general type* if $|mK_X|$ defines a birational map for $m \gg 1$, see (1.30). The *Kodaira dimension* of X, denoted by $\kappa(X)$, is the dimension of the image of $|mK_X|$ for m sufficiently large and divisible.

Notation Commonly Used in Birational Geometry

A *map* or *rational map* is defined on a dense set; it is denoted by $\dashrightarrow$. A *morphism* is everywhere defined; it is denoted by $\to$. A *contraction* is a proper morphism $g\colon X \to Y$ such that $g_*\mathscr{O}_X = \mathscr{O}_Y$.

A map $g\colon X \dashrightarrow Y$ between (possibly reducible) schemes is *birational* if there are nowhere dense closed subsets $Z_X \subset X$ and $Z_Y \subset Y$ such that g restricts to an isomorphism $(X \setminus Z_X) \simeq (Y \setminus Z_Y)$. The smallest such Z_X is the *exceptional set* of g, denoted by $\mathrm{Ex}(g)$. A birational map $g\colon X \dashrightarrow Y$ is *small* if $\mathrm{Ex}(g)$ has codimension ≥ 2 in X.

A *resolution* of X is a proper, birational morphism $p\colon X' \to X$, where X' is nonsingular. X has *rational singularities* if $p_*\mathscr{O}_{X'} = \mathscr{O}_X$ and $R^i p_*\mathscr{O}_{X'} = 0$ for $i > 0$; see Kollár and Mori (1998, Sec.5.1). Rational implies *Cohen–Macaulay,* abbreviated as *CM*; see (10.4).

Let $g\colon X \dashrightarrow Y$ be a birational map defined on the open set $X^\circ \subset X$. For a subscheme $W \subset X$, the closure of $g(W \cap X^\circ) \subset Y$ is the *birational transform,* provided $W \cap X^\circ$ is dense in W. It is denoted by $g_*(W)$

Following the confusion established in the literature, a *divisor on* X is either a prime divisor or a Weil divisor; the context usually makes it clear which one.

We use divisor *over* X to mean a prime divisor on some $\pi\colon X' \to X$ that is birational to X. The *center* of E on X, denoted by $\mathrm{center}_X E$, is (the closure of) $\pi(E) \subset X$.

A *$\mathbb{Z}$-, $\mathbb{Q}$- or $\mathbb{R}$-divisor* (more precisely, Weil $\mathbb{Z}$-, $\mathbb{Q}$- or $\mathbb{R}$-divisor) is a finite linear combinations of prime divisors $\sum a_i D_i$, where $a_i \in \mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$. A divisor is *reduced* if $a_i = 1$ for every i. See Section 4.3 for various versions of divisors (Weil, Cartier, etc.).

A $\mathbb{Z}$- or $\mathbb{Q}$-divisor D on a normal variety is *$\mathbb{Q}$-Cartier* if mD is Cartier for some $m > 0$. (See (11.43) for the $\mathbb{R}$ version.) The smallest $m \in \mathbb{N}$ such that mD is Cartier is called the *Cartier index* or simply *index* of D. On a nonnormal variety Y these notions make sense if Y is nonsingular at the generic points of $\mathrm{Supp}\, D$; we call these *Mumford divisors,* see (4.16.4) and Section 4.8.

The *index* of a variety Y, denoted by $\mathrm{index}(Y)$, is the Cartier index of K_Y.

Linear equivalence of $\mathbb{Z}$-divisors is denoted by $D_1 \sim D_2$. Two $\mathbb{Q}$-divisors are $\mathbb{Q}$-linearly equivalent if $mD_1 \sim mD_2$ for some $m > 0$. It is denoted by $D_1 \sim_{\mathbb{Q}} D_2$. (See (11.43) for the $\mathbb{R}$ version.)

Numerical equivalence of divisors D_i or curves C_i is denoted by $D_1 \equiv D_2$ and $C_1 \equiv C_2$.

The *intersection number* of $\mathbb{R}$-Cartier divisors $D_1, \dots, D_r$ on X with a proper subscheme $Z \subset X$ of dimension r is denoted by $(D_1 \cdots D_r \cdot Z)$ or $(D_1 \cdots D_r)_Z$. We omit Z if $Z = X$, and for self-intersections we use (D^r).

An $\mathbb{R}$-Cartier divisor D (resp. line bundle L) on a proper scheme X is *nef,* if $(D \cdot C) \geq 0$ (resp. $\deg(L|_C) \geq 0$) for every integral curve $C \subset X$.

Let $g\colon X \to S$ be a proper morphism. For a $\mathbb{Q}$-Cartier divisor we use g-ample and *relatively ample* interchangeably; see (11.51) for $\mathbb{R}$-Cartier divisors.

The *rounding down* (resp. *up*) of a real number d is denoted by $\lfloor d \rfloor$ (resp. $\lceil d \rceil$). For a divisor $D = \sum d_i D_i$ we use $\lfloor D \rfloor := \sum \lfloor d_i \rfloor D_i$, where the D_i are distinct, irreducible divisors. The *fractional part* is $\{D\} := D - \lfloor D \rfloor$.

An $\mathbb{R}$-divisor D on a proper, irreducible variety is *big* if $\lfloor mD \rfloor$ defines a birational map for $m \gg 1$.

A *pair* $(X, \Delta = \sum a_i D_i)$ consist of a scheme X and a Weil divisor Δ on it, the coefficients can be in $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$. The divisor part of a pair is frequently called the *boundary* of the pair. (Some authors call Δ a boundary only if $0 \leq a_i \leq 1$ for every i.) When we start with a scheme X and a compactification $X^* \supset X$, frequently $X^* \setminus X$ is also called a *boundary;* this usage is well entrenched for moduli spaces. (Neither agrees with the notion of "boundary" in topology.)

A *simple normal crossing* pair – usually abbreviated as *snc* pair – is a pair (X, D), where X is regular, and at each $p \in X$ there are local coordinates $x_1, \ldots, x_n$ and an open neighborhood $x \in U \subset X$ such that $U \cap \operatorname{Supp} D \subset (x_1 \cdots x_n = 0)$. We also say that D is an *snc divisor.* A scheme Y has *simple normal crossing* singularities if every point $y \in Y$ has an open neighborhood $y \in V \subset Y$ that is isomorphic to an snc divisor.

A *log resolution* of (X, Δ) is a proper, birational morphism $p \colon X' \to X$, where X' is nonsingular and $\operatorname{Supp} \pi^{-1}(\Delta) \cup \operatorname{Ex}(\pi)$ is an snc divisor.

We are mostly interested in proper pairs (X, Δ) with log canonical singularities (11.5). Such a pair is of *general type* if $K_X + \Delta$ is big. In examples, we encounter pairs with $K_X + \Delta \equiv 0$ (called (log)-Calabi–Yau pairs) or with $-(K_X + \Delta)$ ample (called (log)-Fano pairs).

In the literature, "canonical model" can refer to three different notions. We distinguish them as follows. (See Section 11.2 for pairs and for relative versions.)

Given a smooth, proper variety X, its *canonical model* is a proper variety X^{c} that is birational to X, has canonical singularities and ample canonical class.

Given a variety X, its *canonical modification* is a proper, birational morphism $\pi \colon X^{\mathrm{cm}} \to X$ such that X^{cm} has canonical singularities and its canonical class is π-ample.

Given a variety X with resolution $Y \to X$, the canonical model of Y is the *canonical model of resolutions* of X, denoted by X^{cr}. This is independent of Y.

Additional Conventions Used in This Book

These we follow most of the time, but define them at each occurrence.

The normalization of a scheme X is usually denoted by $\bar{X}$ or X^{n}. However, if D is a divisor on X, then usually $\bar{D}$ denotes its preimage in $\bar{X}$. Then $\bar{D}^n$ denotes the normalization of $\bar{D}$. Unfortunately, a bar is also frequently used to denote the compactification of a scheme or moduli space.

Usually, we use $S^\circ \subset S$ to denote an open, dense subset. Then sheaves or divisors on S° are usually indicated by F° or D°. If G is an algebraic group, then G° denotes the identity component.

We write moduli functors in caligraphic and moduli spaces in roman. Thus for stable varieties we have $\mathcal{SV}$ (functor) and SV (moduli space).

Let F, G be quasi-coherent sheaves on a scheme X. Then $\operatorname{Hom}_X(F, G)$ is the set of $\mathscr{O}_X$-linear sheaf homomorphisms (it is also an $H^0(X, \mathscr{O}_X)$-module), and $\mathcal{H}om_X(F, G)$ is the sheaf of $\mathscr{O}_X$-linear sheaf homomorphisms. See (9.34) for the hom-scheme $\mathbf{Hom}_S(F, G)$.

$\operatorname{Mor}_S(X, Y)$ denotes the set of S-morphisms from X to Y, and $\mathbf{Mor}_S(X, Y)$ the scheme that represents the functor $T \mapsto \operatorname{Mor}_S(X \times_S T, Y \times_S T)$ (if it exists); see (8.63). Same conventions for $\operatorname{Isom}_S(X, Y)$ and $\operatorname{Aut}_S(X)$. If X is a proper $\mathbb{C}$-scheme, then one can pretty much identify $\operatorname{Aut}_{\mathbb{C}}(X)$ with $\mathbf{Aut}_{\mathbb{C}}(X)$.

We distinguish the *Picard group* $\operatorname{Pic}(X)$ (as in Hartshorne, 1977), and the *Picard scheme* $\mathbf{Pic}(X)$ (as in Mumford, 1966).

Base change. Given morphisms $f\colon X \to S$ and $q\colon T \to S$, we write the base change diagram as

$$\begin{array}{ccc} X_T & \xrightarrow{q_X} & X \\ {\scriptstyle f_T}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{q} & S. \end{array}$$

Objects obtained by pull-back to X_T are usually denoted either by a subscript T or by q_X^*. The fiber over a point $s \in S$ is denoted by a subscript s. However, we frequently encounter the situation that the fiber product is not the "right" pull-back and needs to be "corrected." Roughly speaking, this happens when the fiber product picks up some embedded subscheme/sheaf, and the "correct" pull-back is the quotient by it.

Thus, for divisors D on X, we let D_T denote the *divisorial pull-back* or *restriction,* which is the divisorial part of $X \times_T D$; see (4.6). We write D_T^{div} if we want to emphasize this (2.73). For coherent sheaves F on X, we frequently use the *hull bull-back*, denoted by F_T^H or $q_X^{[*]}F$; see (3.27).

Brackets are used to denote something naturally associated to an object. We use it to denote the cycle associated to a subscheme (1.3) and the point in the moduli space corresponding to a variety/pair.

The *completion* of a pointed scheme $(x \in X)$ is denoted by $\hat{X}$, or $\hat{X}_x$ if we want to emphasize the point. For $\hat{\mathbb{A}}^n$, the point is assumed the origin, unless otherwise noted. See also (10.52.6).

Numbering

We number everything consecutively. Thus, for example, (2.3) refers to item 3 in Chapter 2. References to sections are given as "Section 2.3." Tertiary numbering is consecutive within items, including lists and formulas. For example, (2.3.2) is subitem 2 in item (2.3), but within (2.3) we may use only (2) as reference.