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Introduction

In the moduli theory of curves, the main objects – *stable curves* – are projective curves C that satisfy two conditions:

- (local) the singularities are nodes, and
- (global) K_C is ample.

Generalizing this, Kollár and Shepherd-Barron (1988) posited that in higher dimensions the objects of the moduli theory – *stable varieties* – are projective varieties X such that

- (local) the singularities are semi-log-canonical, and
- (global) K_X is ample.

The theory of semi-log-canonical singularities is treated in Kollár (2013b). Once the objects of a moduli theory are established, we need to describe the families that we aim to understand. For curves, the answer is clear: flat, projective morphisms whose fibers are stable curves.

By contrast, there are too many flat, projective morphisms whose fibers are stable surfaces; basic numerical invariants are not always constant in such families. The correct notion of *(locally) stable families* of surfaces was defined in Kollár and Shepherd-Barron (1988). We describe these in all dimensions, first for one-parameter families in Chapter 2, and then over an arbitrary base in Chapter 3, where seven equivalent definitions of local stability are given in Definition–Theorem 3.1.

Stable curves with weighted points also appeared in many contexts, and, correspondingly, the general objects in higher dimensions are pairs (X, Δ), where X is a variety and $\Delta = \sum a_i D_i$ is a formal linear combination of divisors with rational or real coefficients. Such a pair (X, Δ) is *stable* iff

- (local) the singularities are semi-log-canonical, and
- (global) $K_X + \Delta$ is ample.

The main aim of this book is to complete the moduli theory of stable pairs in characteristic 0.

Defining the right notion of *(locally) stable families of pairs* turned out to be very challenging. The reason is that the divisorial part Δ is not necessarily flat

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over the base. Flatness was built into the foundations of algebraic geometry by Grothendieck, and many new results had to be developed.

Our solution goes back to the works of Cayley (1860, 1862), who associated a divisor in $\mathrm{Gr}(1,3)$ – the Grassmannian of lines in $\mathbb{P}^3$ – to any space curve. More generally, given any subvariety $X^d \subset \mathbb{P}^n$ and a divisor D on X, there is a Cayley hypersurface $\mathrm{Ca}(D) \subset \mathrm{Gr}(n-d, n)$. We declare a family of divisors $\{D_s : s \in S\}$ *C-flat* if the corresponding Cayley hypersurfaces $\{\mathrm{Ca}(D_s) : s \in S\}$ form a flat family. This turns out to work very well over reduced base schemes, leading to a complete moduli theory of stable families of pairs over such bases. This is done in Chapter 4. For the rest of the book, the key result is Theorem 4.76, which constructs the universal family of C-flat Mumford divisors over an arbitrary base. While C-flatness is defined using a projective embedding, it is independent of it over reduced bases, but most likely not in general.

Chapter 5 contains numerical criteria for various fiber-wise constructions to fit together into a flat family. For moduli theory the most important result is Theorem 5.1: a flat, projective morphism $f : X \to S$ is stable iff the fibers are stable and the volume of the fibers $(K^n_{X_s})$ is locally constant on S.

Chapter 6 discusses several special cases where flatness is the right notion for the divisor part of a family of stable pairs. This includes all the pairs $(X, \Delta := \sum a_i D_i)$ with $a_i > \frac{1}{2}$ for every i; see Theorem 6.29.

The technical core of the book is Chapter 7. We develop the notion of K-flatness, which is a version of C-flatness that is independent of the projective embedding; see Definition 7.1. It has surprisingly many good properties, listed in Theorems 7.3–7.5. We believe that this is the "correct" concept for moduli purposes. However, the proofs are rather nuts-and-bolts; a more conceptual approach would be very desirable.

All of these methods and results are put together in Chapter 8 to arrive at Theorem 8.1, which is the main result of the book: The notion of Kollár–Shepherd-Barron–Alexeev stability for families of stable pairs yields a good moduli theory, with projective coarse moduli spaces.

Section 8.8 discusses problems that complicate the moduli theory of pairs in positive characteristic; some of these appear quite challenging.

The remaining chapters are devoted to auxiliary results. Chapter 9 discusses hulls and husks, a generalization of quot schemes, that was developed to suit the needs of higher dimensional moduli theory. Chapter 10 collects sundry results for which we could not find good references, while Chapter 11 summarizes the key concepts and theorems of Kollár (2013b), as well as the main results of the minimal model program that we need.

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1

History of Moduli Problems

The moduli spaces of smooth or stable projective curves of genus $g \geq 2$ are, quite possibly, the most studied of all algebraic varieties.

The aim of this book is to generalize the moduli theory of curves to surfaces and to higher dimensional varieties. In this chapter, we aim to outline how this is done, and, more importantly, to explain why the answer for surfaces is much more complicated than for curves. On the positive side, once we get the moduli theory of surfaces right, the higher dimensional theory works the same.

Section 1.1 is a quick review of the history of moduli problems, culminating in an outline of the basic moduli theory of curves. A'Campo et al. (2016) is a very good overview. Reading some of the early works on moduli, including Riemann, Cayley, Klein, Hilbert, Siegel, Teichmüller, Weil, Grothendieck, and Mumford gives an understanding of how the modern theory relates to the earlier works. See Kollár (2021b) for an account that emphasizes the historical connections.

In Section 1.2, we outline how the theory should unfold for higher dimensional varieties. Details of going from curves to higher dimensions are given in the next two sections. Section 1.3 introduces canonical models, which are the basic objects of moduli theory in higher dimensions. Starting from stable curves, Section 1.4 leads up to the definition of stable varieties, their higher dimensional analogs. Then we show, by a series of examples, why flat families of stable varieties are *not* the correct higher dimensional analogs of flat families of stable curves. Finding the correct replacement has been one of the main difficulties of the whole theory.

While the moduli theory of curves serves as our guideline, it also has many good properties that do not generalize. Sections 1.5–1.8 are devoted to examples that show what can go wrong with moduli theory in general, or with stable varieties in particular.

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First, in Section 1.5, we show that the simple combinatorial recipe of going from a nodal curve to a stable curve has no analog for surfaces. Next we give a collection of examples showing how easy it is to end up with rather horrible moduli problems. Hypersurfaces and other interesting examples are discussed in Section 1.6, as are alternative compactifications of the moduli of curves in Section 1.7. Section 1.8 illustrates the differences between fine and coarse moduli spaces.

Two major approaches to moduli – the geometric invariant theory of Mumford and the Hodge theory of Griffiths – are mostly absent from this book. Both of these are very powerful, and give a lot of information in the cases when they apply. They each deserve a full, updated treatment of their own. However, so far neither gave a full description of the moduli of surfaces, much less of higher dimensional varieties. It would be very interesting to develop a synthesis of the three methods and gain better understanding in the future.

1.1 Riemann, Cayley, Hilbert, and Mumford

Let **V** be a "reasonable" class of objects in algebraic geometry, for instance, **V** could be all subvarieties of $\mathbb{P}^n$, all coherent sheaves on $\mathbb{P}^n$, all smooth curves or all projective varieties. The aim of the theory of moduli is to understand all "reasonable" families of objects in **V**, and to construct an algebraic variety (or scheme, or algebraic space) whose points are in "natural" one-to-one correspondence with the objects in **V**. If such a variety exists, we call it the *moduli space* of **V**, and denote it by $\mathrm{M}_{\mathbf{V}}$. The simplest, classical examples are given by the theory of linear systems and families of linear systems.

1.1 (Linear systems) Let X be a smooth, projective variety over an algebraically closed field k and L a line bundle on X. The corresponding linear system is

$$\mathcal{L}in\mathcal{S}ys(X, L) = \{\text{effective divisors } D \text{ such that } \mathscr{O}_X(D) \simeq L\}.$$

The objects in $\mathcal{L}in\mathcal{S}ys(X, L)$ are in natural one-to-one correspondence with the points of the projective space $\mathbb{P}(H^0(X, L)^\vee)$ which is traditionally denoted by $|L|$. (We follow the Grothendieck convention for $\mathbb{P}$ as in Hartshorne (1977, sec.II.7).) Thus, for every effective divisor D such that $\mathscr{O}_X(D) \simeq L$, there is a unique point $[D] \in |L|$.

Moreover, this correspondence between divisors and points is given by a universal family of divisors over $|L|$. That is, there is an effective Cartier divisor $\mathrm{Univ}_L \subset |L| \times X$ with projection $\pi\colon \mathrm{Univ}_L \to |L|$ such that $\pi^{-1}[D] = D$ for every effective divisor D linearly equivalent to L.

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The classical literature never differentiates between the linear system as a set and the linear system as a projective space. There are, indeed, few reasons to distinguish them as long as we work over a fixed base field k. If, however, we pass to a field extension $K \supset k$, the advantages of viewing $|L|$ as a k-variety appear. For any $K \supset k$, the set of effective divisors D defined over K such that $\mathscr{O}_X(D) \simeq L$ corresponds to the K-points of $|L|$. Thus the scheme-theoretic version automatically gives the right answer over every field.

1.2 (Jacobians of curves) Let C be a smooth projective curve (or Riemann surface) of genus g. As discovered by Abel and Jacobi, there is a variety $\mathrm{Jac}^\circ(C)$ of dimension g whose points are in natural one-to-one correspondence with degree 0 line bundles on C. As before, the correspondence is given by a universal line bundle $L^{\mathrm{univ}} \to C \times \mathrm{Jac}^\circ(C)$, called the Poincaré bundle. That is, for any point $p \in \mathrm{Jac}^\circ(C)$, the restriction of L^{univ} to $C \times \{p\}$ is the degree 0 line bundle corresponding to p.

Unlike in (1.1), the universal line bundle L^{univ} is not unique (and need not exist if the base field is not algebraically closed). This has to do with the fact that while an automorphism of the pair $D \subset X$ that is trivial on X is also trivial on D, any line bundle $L \to C$ has automorphisms that are trivial on C: we can multiply every fiber of L by the same nonzero constant.

1.3 (Cayley forms and Chow varieties) Cayley (1860, 1862) developed a method to associate a hypersurface in the Grassmannian $\mathrm{Gr}(\mathbb{P}^1, \mathbb{P}^3)$ to a curve in $\mathbb{P}^3$. The resulting moduli spaces have been used, but did not seem to have acquired a name. Chow understood how to deal with reducible and multiple varieties, and proved that one gets a projective moduli space; see Chow and van der Waerden (1937). The name *Chow variety* seems standard, we use Cayley–Chow for the correspondence that was discovered by Cayley. See Section 3.1 for an outline and Kollár (1996, secs.I.3–4) for a modern treatment.

Let k be an algebraically closed field and X a normal, projective k-variety. Fix a natural number m. An *m-cycle* on X is a finite, formal linear combination $\sum a_i Z_i$ where the Z_i are irreducible, reduced subvarieties of dimension m and $a_i \in \mathbb{Z}$. We usually assume tacitly that all the Z_i are distinct. An m-cycle is called *effective* if $a_i \geq 0$ for every i.

The points of the *Chow variety* $\mathrm{Chow}_m(X)$ are in "natural" one-to-one correspondence with the set of effective m-cycles on X. (Since we did not fix the degree of the cycles, $\mathrm{Chow}_m(X)$ is not actually a variety, but a countable disjoint union of projective, reduced k-schemes.) The point of $\mathrm{Chow}_m(X)$ corresponding to a cycle $Z = \sum a_i Z_i$ is also usually denoted by $[Z]$.

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As for linear systems, it is best to describe the "natural correspondence" by a universal family. The situation is, however, more complicated than before.

There is a family (or rather an effective cycle) $\mathrm{Univ}_m(X)$ on $\mathrm{Chow}_m(X) \times X$ with projection $\pi\colon \mathrm{Univ}_m(X) \to \mathrm{Chow}_m(X)$ such that for every effective m-cycle $Z = \sum a_i Z_i$,

(1.3.1) the support of $\pi^{-1}[Z]$ is $\cup_i Z_i$, and

(1.3.2) the fundamental cycle (4.61.1) of $\pi^{-1}[Z]$ equals Z if $a_i = 1$ for every i.

If the characteristic of k is 0, then the only problem in (2) is a clash between the traditional cycle-theoretic definition of the Chow variety and the scheme-theoretic definition of the fiber, but in positive characteristic the situation is more problematic; see Kollár (1996, secs.I.3–4).

An example of a "perfect" moduli problem is the theory of *Hilbert schemes*, introduced in Grothendieck (1962, lect.IV). See Mumford (1966), (Kollár, 1996, I.1–2) or Sernesi (2006, sec.4.3) or Section 3.1 for a summary.

1.4 (Hilbert schemes) Let k be an algebraically closed field and X a projective k-scheme. Set

$$\mathcal{H}ilb(X) = \{\text{closed subschemes of } X\}.$$

Then there is a k-scheme $\mathrm{Hilb}(X)$, called the *Hilbert scheme* of X, whose points are in a "natural" one-to-one correspondence with closed subschemes of X. The point of $\mathrm{Hilb}(X)$ corresponding to a subscheme $Y \subset X$ is frequently denoted by $[Y]$. There is a universal family $\mathrm{Univ}(X) \subset \mathrm{Hilb}(X) \times X$ such that

(1.4.1) the first projection $\pi\colon \mathrm{Univ}(X) \to \mathrm{Hilb}(X)$ is flat, and

(1.4.2) $\pi^{-1}[Y] = Y$ for every closed subscheme $Y \subset X$.

The beauty of the Hilbert scheme is that it describes not just subschemes, but all flat families of subschemes as well. To see what this means, note that for any morphism $g\colon T \to \mathrm{Hilb}(X)$, by pull-back we obtain a flat family of subschemes $T \times_{\mathrm{Hilb}(X)} \mathrm{Univ}(X) \subset T \times X$. It turns out that every family is obtained this way:

(1.4.3) For every T and closed subscheme $Z \subset T \times X$ that is flat over T, there is a unique $g_Z\colon T \to \mathrm{Hilb}(X)$ such that $Z = T \times_{\mathrm{Hilb}(X)} \mathrm{Univ}(X)$.

This takes us to the functorial approach to moduli problems.

1.5 (Hilbert functor and Hilbert scheme) Let $X \to S$ be a morphism of schemes. Define the *Hilbert functor* of X/S as a functor that associates to a scheme $T \to S$ the set

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$$\mathcal{H}ilb_{X/S}(T) = \{\text{subschemes } Z \subset T \times_S X \text{ that are flat and proper over } T\}.$$

The basic existence theorem of Hilbert schemes then says that, if $X \to S$ is quasi-projective, there is a scheme $\operatorname{Hilb}_{X/S}$ such that for any S scheme T,

$$\mathcal{H}ilb_{X/S}(T) = \operatorname{Mor}_S(T, \operatorname{Hilb}_{X/S}).$$

Moreover, there is a universal family $\pi\colon \operatorname{Univ}_{X/S} \to \operatorname{Hilb}_{X/S}$ such that the above isomorphism is given by pulling back the universal family.

We can summarize these results as follows:

Principle 1.6 $\pi\colon \operatorname{Univ}_{X/S} \to \operatorname{Hilb}_{X/S}$ *contains all the information about proper, flat families of subschemes of X/S, in the most succinct way.*

This example leads us to a general definition:

Definition 1.7 (Fine moduli spaces) Let $\mathbf{V}$ be a "reasonable" class of projective varieties (or schemes, or sheaves, or ...). In practice "reasonable" may mean several restrictions, but for the definition we only need the following weak assumption:

(1.7.1) Let $K \supset k$ be a field extension. Then a k-variety X_k is in $\mathbf{V}$ iff $X_K := X_k \times_{\operatorname{Spec} k} \operatorname{Spec} K$ is in $\mathbf{V}$.

Following (1.5), define the corresponding moduli functor that associates to a scheme T the set

$$\mathcal{V}arieties_{\mathbf{V}}(T) := \left\{ \begin{array}{c} \text{Flat families } X \to T \text{ such that} \\ \text{every fiber is in } \mathbf{V}, \\ \text{modulo isomorphisms over } T. \end{array} \right\} \qquad (1.7.2)$$

We say that a scheme $\operatorname{Moduli}_{\mathbf{V}}$ is a *fine moduli space* for the functor $\mathcal{V}arieties_{\mathbf{V}}$, if the following holds:

(1.7.3) For every scheme T, pulling back gives an equality

$$\mathcal{V}arieties_{\mathbf{V}}(T) = \operatorname{Mor}(T, \operatorname{Moduli}_{\mathbf{V}}).$$

Applying the definition to $T = \operatorname{Moduli}_{\mathbf{V}}$ gives a universal family $u\colon \operatorname{Univ}_{\mathbf{V}} \to \operatorname{Moduli}_{\mathbf{V}}$. Setting $T = \operatorname{Spec} K$, where K is a field, we see that the K-points of $\operatorname{Moduli}_{\mathbf{V}}$ correspond to the K-isomorphism classes of objects in $\mathbf{V}$.

We consider the existence of a fine moduli space as the ideal possibility. Unfortunately, it is rarely achieved.

Next we see what happens with the simplest case, for smooth curves.

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1.8 (Moduli functor and moduli space of smooth curves) Following (1.7) we define the moduli functor of smooth curves of genus g as

$$\mathit{Curves}_g(T) := \left\{ \begin{array}{c} \text{Smooth, proper families } S \to T, \\ \text{every fiber is a curve of genus } g, \\ \text{modulo isomorphisms over } T. \end{array} \right\}$$

It turns out that there is no fine moduli space for curves of genus g. Every curve C with nontrivial automorphisms causes problems; there cannot be any point $[C]$ corresponding to it in a fine moduli space (see Section 1.8).

It was gradually understood that there is some kind of an object, denoted by M_g, and called the *coarse moduli space* (or simply *moduli space*) of curves of genus g, that comes close to being a fine moduli space.

For elliptic curves, we get $\mathrm{M}_1 \simeq \mathbb{A}^1$, and the moduli map is given by the j-invariant, as was known to Dedekind and Klein; see Klein and Fricke (1892). They also knew that there is no universal family over M_1. The theory of abelian integrals due to Abel, Jacobi, and Riemann does the same for all curves, though in this case a clear moduli-theoretic interpretation seems to have been done only later; see the historical sketch at the end of Shafarevich (1974), Siegel (1969, chap.4), or Griffiths and Harris (1978, chap.2) for modern treatments. For smooth plane curves, and more generally for smooth hypersurfaces in any dimension, the invariant theory of Hilbert produces coarse moduli spaces. Still, a precise definition and proof of existence of M_g appeared only in Teichmüller (1944) in the analytic case and in Mumford (1965) in the algebraic case. See A'Campo et al. (2016) or Kollár (2021b) for historical accounts.

1.9 (Coarse moduli spaces) Mumford (1965)

As in (1.7), let $\mathbf{V}$ be a "reasonable" class. When there is no fine moduli space, we still can ask for a scheme that best approximates its properties.

We look for schemes M for which there is a natural transformation

$$T_M \colon \mathcal{V}\mathit{arieties}_g(*) \longrightarrow \mathrm{Mor}(*, M).$$

Such schemes certainly exist: for instance, if we work over a field k, then we can take $M = \operatorname{Spec} k$. All schemes M for which T_M exists form an inverse system which is closed under fiber products. Thus, as long as we are not unlucky, there is a universal (or largest) scheme with this property. Though it is not usually done, it should be called the *categorical moduli space*.

This object can be rather useless in general. For instance, fix n, d and let $\mathbf{H}_{n,d}$ be the class of all hypersurfaces of degree d in $\mathbb{P}_k^{n+1}$, up to isomorphisms. We see in (1.56) that a categorical moduli space exists and it is $\operatorname{Spec} k$.

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To get something more like a fine moduli space, we require that it give a one-to-one parametrization, at least set theoretically. Thus we say that a scheme $\mathrm{Moduli}_{\mathbf{V}}$ is a *coarse moduli space* for **V** if the following hold:

(1.9.1) there is a natural transformation of functors

$$\mathrm{ModMap}\colon \mathcal{V}arieties_{\mathbf{V}}(*) \longrightarrow \mathrm{Mor}(*, \mathrm{Moduli}_{\mathbf{V}}),$$

(1.9.2) $\mathrm{Moduli}_{\mathbf{V}}$ is universal satisfying (1), and

(1.9.3) for any algebraically closed field $K \supset k$, we get a bijection

$$\mathrm{ModMap}\colon \mathcal{V}arieties_{\mathbf{V}}(\operatorname{Spec} K) \stackrel{\simeq}{\longrightarrow} \mathrm{Mor}(\operatorname{Spec} K, \mathrm{Moduli}_{\mathbf{V}}) = \mathrm{Moduli}_{\mathbf{V}}(K).$$

1.10 (Moduli functors versus moduli spaces) While much of the early work on moduli, especially since Mumford (1965), put the emphasis on the construction of fine or coarse moduli spaces, recently the focus shifted toward the study of the families of varieties, that is, toward moduli functors and moduli stacks. The main task is to understand all "reasonable" families. Once this is done, the existence of a coarse moduli space should be nearly automatic. The coarse moduli space is not the fundamental object any longer, rather it is only a convenient way to keep track of certain information that is only latent in the moduli functor or stack.

1.11 (Compactifying M_g) While the basic theory of algebraic geometry is local, that is, it concerns affine varieties, most really interesting and important objects in algebraic geometry and its applications are global, that is, projective or at least proper.

The moduli spaces M_g are not compact, in fact the moduli functor of smooth curves discussed so far has a definitely local flavor. Most naturally occurring smooth families of curves live over affine schemes, and it is not obvious how to write down any family of smooth curves over a projective base. For many reasons it is useful to find geometrically meaningful compactifications of M_g. The answer to this situation is to allow not just smooth curves, but also certain singular curves in our families.

Concentrating on one-parameter families, we have the following:

Question 1.11.1 Let B be a smooth curve, $B^\circ \subset B$ an open subset, and $\pi^\circ\colon S^\circ \to B^\circ$ a smooth family of genus g curves. Is there a "natural" extension

$$\begin{array}{ccc} S^\circ & \hookrightarrow & S \\ {\scriptstyle \pi^\circ}\downarrow & & \downarrow{\scriptstyle \pi} \\ B^\circ & \hookrightarrow & B, \end{array}$$

where $\pi\colon S \to B$ is a flat family of (possibly singular) curves?

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There is no reason to think that there is a unique such extension. Deligne and Mumford (1969) construct one after a base change $B' \to B$, and by now it is hard to imagine a time when their choice was not the "obviously best" solution. We review their definition next. In Section 1.6 we see, by examples, why this concept has not been so obvious.

Definition 1.12 (Stable curve) A *stable curve* over an algebraically closed field k is a proper, geometrically connected k-curve C such that

(Local property) the only singularities of C are ordinary nodes, and

(Global property) the canonical class K_C is ample.

A *stable curve* over a scheme T is a flat, proper morphism $\pi: S \to T$ such that every geometric fiber of π is a stable curve. (The arithmetic genus of the fibers is a locally constant function on T, but we usually also tacitly assume that it is constant.) The moduli functor of stable curves of genus g is

$$\overline{\mathit{Curves}}_g(T) := \left\{ \begin{array}{c} \text{Stable curves of genus } g \text{ over } T, \\ \text{modulo isomorphisms over } T. \end{array} \right\}$$

Theorem 1.13 Deligne and Mumford (1969) *For every $g \geq 2$, the moduli functor of stable curves of genus g has a coarse moduli space $\overline{\mathrm{M}}_g$. Moreover, $\overline{\mathrm{M}}_g$ is projective, normal, has only quotient singularities, and contains M_g as an open dense subset.*

$\overline{\mathrm{M}}_g$ has a rich and intriguing geometry, which is related to major questions in many branches of mathematics and physics; see Farkas and Morrison (2013) for a collection of surveys and Pandharipande (2018a,b) for overviews.

1.2 Moduli for Varieties of General Type

The aim of this book is to use the moduli of stable curves as a guideline, and develop a moduli theory for varieties of general type (1.30). (See (1.22) for some comments on the nongeneral type cases.)

Here we outline the main steps of the plan with some comments. Most of the rest of the book is then devoted to accomplishing these goals.

Step 1.14 (Higher dimensional analogs of smooth curves) It has been understood since the beginnings of the theory of surfaces that, for surfaces of Kodaira dimension ≥ 0 (p.xiv), the correct moduli theory should be birational, not biregular. That is, the points of the moduli space should correspond not to *isomorphism* classes of surfaces, but to *birational* equivalence classes of surfaces. There are two ways to deal with this problem.

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