1 Introduction

Mathematical anti-realism is (somewhat roughly) the view that our mathematical theories don't provide true descriptions of mathematical objects (i.e., things like numbers and sets) because there are no such things as mathematical objects. The reason there are no mathematical objects, on this view, is that (a) if there were mathematical objects, they would be *abstract* objects – that is, nonphysical, nonmental, nonspatiotemporal objects – and (b) there are no abstract objects.

My central aim in this Element is to articulate what I think is an underappreciated problem for mathematical anti-realism – a problem having to do with the truthmaking of modal claims – and to develop and defend what I think is the only plausible solution to that problem.

In the first part of this Element, I'll provide a more thorough and precise articulation of mathematical anti-realism, I'll explain how I think that view should be developed, and I'll respond to a number of objections that you might raise against that view. But at the end of the first part, another objection will emerge, an objection based on the fact that (a) mathematical anti-realists need to commit to the truth of certain kinds of *modal* claims (i.e., *possibility* claims and/or *necessity* claims and/or *counterfactuals*), and (b) it's not clear that the truth of these modal claims is compatible with mathematical anti-realism. In the second part of this Element, I'll run through the various strategies that anti-realists might pursue in trying to solve this modal-truth problem – that is, the problem of explaining how modal claims of the relevant kind could be true, given that mathematical anti-realism is also true – and I'll argue that there's only one viable view available to anti-realists here. I'll call this view *modal nothingism*, and I'll argue at the end of the second part that modal nothingism is true and that mathematical anti-realists can use this view to block the modal-truth objection to their view.

The first part of this Element consists of Sections 2–4. In Section 2, I'll do three things: I'll define mathematical realism and anti-realism; I'll distinguish three different versions of anti-realism, namely, paraphrase nominalism, deflationary-truth nominalism, and mathematical error theory; and I'll argue against paraphrase nominalism and deflationary-truth nominalism.

In Section 3, I'll respond to four objections to error theory. I'll spend most of my time responding to the worry that error theorists can't account for the factualness and objectivity of mathematics. After that, I'll respond to three more objections to error theory – a Moorean objection, a Lewisian objection, and a Quine-Putnam indispensability objection. During the course of the discussion, it will become clear that in order to respond to these objections, error theorists need to commit to the truth of certain kinds of *modal* claims.

In particular, I'll argue that they need to commit to the truth of either counterfactuals like

> [CF] If there had actually existed a plenitudinous realm of abstract objects, then it would have been the case that 3 was prime,

or necessitarian conditionals like

> [N] Necessarily, if there exists a plenitudinous realm of abstract objects, then 3 is prime.

Finally, at the end of Section 3, I'll point out that this creates a problem for error theory. For given that error theorists deny that there are abstract objects, it's not clear that they have access to any plausible account of how modal sentences like [CF] and [N] could be true.

In Section 4, I'll argue that, like error theorists, paraphrase nominalists and deflationary-truth nominalists need to commit to the truth of modal sentences like [CF] or [N], and so – again, like error theorists – they encounter the worry that they don't have access to any plausible account of how sentences like [CF] and [N] could be true, given that there are no such things as abstract objects.

The second part of this Element consists of Sections 5–8. In Section 5, I'll do two things. First, I'll introduce the two most prominent views of the semantics of ordinary modal sentences, namely, *the possible-worlds analysis* (which says, roughly, that ordinary modal sentences are claims about possible worlds) and *modal primitivism* (which says, roughly, that ordinary modal sentences involve primitive modal operators). Second, I'll introduce a principle that, prima facie, seems extremely plausible – namely, that for any true sentence S, *there's something about reality that makes S true* – and I'll point out that (a) given the plausibility of this principle, mathematical anti-realists owe an account of what it is about reality that makes modal sentences like [CF] and [N] true, and (b) given that anti-realists don't believe in abstract objects, it's not clear that they have access to any tenable view of what makes these modal sentences true.

In Section 6, I'll argue that, if they want to, mathematical anti-realists can endorse the possible-worlds analysis of ordinary modal discourse – if they also endorse a *modal error theory*; but I'll also argue that if they do this, then in order to account for the objectivity and factualness of our modal and mathematical discourse, they'll have to introduce novel primitive modal operators (i.e., primitive modal operators that aren't part of ordinary language), and moreover, they'll have to answer the truthmaking question for sentences involving these novel primitive modal operators. Thus, even if anti-realists endorse the possible-worlds analysis of ordinary modal discourse, they'll end up facing a modal truthmaking problem that's essentially equivalent to the modal truthmaking problem that they face if they endorse modal primitivism.

In Section 7, I'll turn to the question of whether mathematical anti-realists can solve the modal truthmaking problem if they endorse modal primitivism, and I'll argue that they don't seem to have any viable options here. More specifically, I'll argue that anti-realists don't have access to any plausible view of what makes ordinary modal sentences true. I'll do this by running through a number of options – including conventionalist views, essentialist views, and potentiality views – and arguing that none of the available views gives anti-realists what they need.

Finally, in Section 8, I'll argue that there's a way out of this problem for anti-realists. I'll do this by arguing against the assumption that every true sentence is made true by reality. More specifically, I'll argue that while our modal sentences are true – indeed, substantively and objectively true – *there's nothing about reality that makes them true*. I'll call this view *modal nothingism*, and in Section 8, I'll argue that it's true and that it gives mathematical anti-realists a solution to the modal truthmaking problem with their view.

MATHEMATICAL ANTI-REALISM

2 What Is Mathematical Anti-Realism?

2.1 Opening Remarks

In order to understand mathematical anti-realism, we first need to understand mathematical realism – because the former can be defined as simply the negation of the latter. Thus, I'll begin, in Section 2.2, by discussing mathematical realism (in particular, I'll define realism and explain why one might be attracted to the view, why one might be dissatisfied with it, and why the only tenable versions of realism involve a commitment to *platonism*). Then in Section 2.3, I'll define mathematical anti-realism, and I'll distinguish three different versions of that view – namely, paraphrase nominalism, deflationary-truth nominalism, and error theory. Finally, in Section 2.4, I'll argue against paraphrase nominalism and deflationary-truth nominalism.

2.2 Mathematical Realism

2.2.1 Mathematical Realism Defined

The best way to understand mathematical realism is to begin by thinking about mathematical theories (like Peano Arithmetic and Zermelo–Frankel set theory) and simple mathematical sentences (like '3 is prime'). Prima facie, '3 is prime' seems to be a true claim about a certain object, namely, the number 3; '3 is prime' seems analogous in this way to, for instance, 'Mars is round.' Just as the latter tells us that a certain object (Mars) has a certain property (roundness), so the former tells us that the object 3 has the property of being prime. And – again, prima facie – it seems that our mathematical theories give us a bunch of true claims about a bunch of objects; for example, Peano Arithmetic tells us that every natural number has a successor, and that there are infinitely many primes, and so on.

Mathematical realism is the view that these appearances (about theories like Peano Arithmetic and sentences like '3 is prime') are indeed true. More precisely, realism is the view that (a) there exist mathematical objects (e.g., numbers and sets), and (b) our mathematical theories (which include ordinary sentences like '3 is prime') provide true descriptions of these objects.

2.2.2 The Argument for Mathematical Realism

Consider the following argument:

[1] Mathematical sentences like '3 is prime' should be read at face value and, hence, as making straightforward claims about the nature of certain specific objects; for example, '3 is prime' should be read as being of the form '*Fa*' and, hence, as making a straightforward claim about the nature of the number 3. But

[2] If sentences like '3 is prime' should be read at face value, and if moreover they're true, then there must actually exist objects of the kinds that they're about; for example, if '3 is prime' makes a straightforward claim about the nature of the number 3, and if this sentence is literally true, then there must actually exist such a thing as the number 3. Moreover,

[3] Mathematical sentences like '3 is prime' *are* true. Therefore, from [1]–[3], it follows that

[4] There actually exist objects that mathematical sentences like '3 is prime' are about, and these sentences provide true descriptions of those objects; for example, the number 3 actually exists, and '3 is prime' provides a true description of that object.[1]

This argument is valid, and [4] is essentially equivalent to mathematical realism, and so if [1]–[3] are all true, then mathematical realism is also true. But, prima facie, it seems that [1]–[3] *are* true (and I'll articulate some arguments for these three premises in Sections 2.4, 3.2.1, and 3.4), and so it seems that mathematical realism is true.

1 Arguments of this general kind have been given by many philosophers, most notably, Frege (1954, 1964).

2.2.3 Platonism as the Only Tenable Version of Mathematical Realism

We can define the following three versions of mathematical realism:

> *Physicalism (about mathematics)*: Our mathematical sentences and theories provide true descriptions of physical objects.
>
> *Psychologism (about mathematics)*: Our mathematical sentences and theories provide true descriptions of mental objects, presumably ideas in our heads.
>
> *Platonism (about mathematics)*: Our mathematical sentences and theories provide true descriptions of *abstract* objects (i.e., objects that are nonphysical, nonmental, nonspatiotemporal, unextended, and acausal).[2]

The most popular and prominent version of realism – and by far the *best* version, in my opinion – is platonism. This might be surprising. For the existence of abstract objects is extremely controversial – lots of people don't believe in them at all – and the existence of physical objects and mental objects (e.g., ideas in our heads) is relatively *un*controversial. So you might think that physicalism and psychologism are superior to platonism because they're metaphysically uncontroversial.

But there are very strong arguments against physicalism and psychologism (about mathematics). I won't go into any depth on these arguments here, but, very quickly, one argument against physicalism and psychologism is based on the claim that these two views falsely imply that the truth of our mathematical theories requires the existence of a gigantic infinity of physical or mental objects. Consider, for example, the following sentence (which is basically an informal version of Cantor's theorem):

> [CT] There are infinitely many transfinite cardinals that keep getting bigger and bigger without end.

Given this, we can argue against physicalism (or psychologism) as follows:

> If physicalism (or psychologism) were true, then the truth of [CT] would require the existence of a massive infinity of physical (or mental) objects. But (a) it seems likely that there just aren't that many physical (or mental) objects in the entire universe; and, more importantly, (b) the truth of [CT] actually *doesn't* depend on the existence of a massive infinity of physical (or mental) objects. If you rejected [CT] on the grounds that there aren't enough physical (or mental) objects in the entire universe to make [CT] true, that would just

[2] Platonist views have been defended by, among others, Frege (1953, 1964), Russell (1903), Quine (1961a, 1961b), Gödel (1964), Putnam (1971), Resnik (1997), Shapiro (1997), me (Balaguer 1998), Zalta (1999), Colyvan (2001), and on some interpretations, Maddy (1990). Physicalist views – or at least views with physicalist leanings – have been endorsed by Mill (1843), Kitcher (1984), and on some interpretations, Maddy (1990). Finally, views with psychologistic leanings have been endorsed by Brouwer (1983a, 1983b) and Heyting (1956).

> show that you don't understand what [CT] *says*. In standard mathematical settings, [CT] just isn't a claim about physical (or mental) objects; it doesn't imply that there are *any* physical (or mental) objects, let alone a huge infinity of them. And this suggests that physicalism (and psychologism) are just false.

This argument is very quick, and there are various ways in which advocates of physicalism and psychologism might respond. But I don't think any of these responses succeed, and what's more, I think there are other good arguments against physicalism and psychologism. (For more arguments here, see Frege [1954, 1964], Resnik [1980], and me [Balaguer 1998, 2014].) But I won't run through the details of any of this here. Instead, I'll just assume that there are good arguments against physicalism and psychologism (about mathematics) – and, hence, that platonism is the only tenable version of realism. In other words, I'll assume – as is fairly standard in the literature these days – that if mathematical realism is true, then platonism is also true, so that our only viable options are platonism and anti-realism.

2.2.4 Why You Might Be Unhappy with Platonism (and, Hence, Attracted to Anti-Realism)

Given that the only tenable realist option is platonism, you might be inclined to reject that view. For you might simply not believe in abstract objects. Why? Well, not to put too fine a point on it, because you might think that abstract objects would be *metaphysically weird*, or *supernatural*, if they existed. Abstract objects are supposed to be wholly nonphysical and nonmental; they're not supposed to be *made* of anything; they're not supposed to be located anywhere (they're supposed to exist but not in space or time); and they're supposed to be wholly unextended and causally inert. This is a lot to swallow. And, again, it seems metaphysically supernatural; prima facie, believing in abstract objects seems to be like believing in ghosts or genies. We seem to have good reason *not* to believe in such things. We seem to have good reason to endorse a materialistic worldview according to which everything is physical. And so we seem to have good reason not to believe in abstract objects.

I don't know whether these remarks count as an *argument*. If not, and if you want an argument against platonism, you can use the widely discussed *epistemological* argument (developed by Benacerraf [1973] and Field [1989]). Here's a very simple version of this argument:

> Since abstract objects would exist outside of space and time (if they existed), and since we humans exist entirely within space and time, we could never acquire any knowledge of abstract objects. But platonism implies that mathematical knowledge *is* knowledge of abstract objects, and so platonism is incompatible with the fact that we humans *do have* mathematical knowledge.

There's a lot to say about whether this argument is good – that is, whether it succeeds in refuting platonism and motivating anti-realism[3] – but I won't say any more about this here because my aim in this Element is not to provide a positive argument for anti-realism. My aim is rather to defend anti-realism against the objections to that view.

2.3 Three Versions of Mathematical Anti-Realism

We can define mathematical anti-realism as just the negation of mathematical realism. So anti-realism is the view that it's not the case that our mathematical theories provide true descriptions of really existing mathematical objects. More roughly, but also more simply, anti-realism is the view that there are no such things as mathematical objects like numbers and sets and so on.[4]

There are numerous versions of anti-realism, and we can get a handle on these views by thinking about the argument for realism that I gave in Section 2.2.2 (i.e., the argument in [1]–[4]). That argument contains three premises – [1], [2], and [3] – and anti-realists have to reject one of these premises (because, together, they entail realism). And what we'll now see is that each of these three strategies of response leads to a different version of anti-realism. If we reject [1], we're led to *paraphrase nominalism*; if we reject [2], we're led to *deflationary-truth nominalism*; and if we reject [3], we're led to *error theory.* I will now articulate these three views.

2.3.1 Paraphrase Nominalism and the Rejection of Premise [1]

Paraphrase nominalism is the view that ordinary mathematical sentences should not be read at face value (and, in particular, that these sentences should not be read as making claims about objects). So, for example, '3 is prime' should not be read as being of the form '*Fa*' (and it should not be read as making a claim about the number 3); and 'There are infinitely many prime numbers' should not be read as asserting the actual existence of infinitely many objects; and so on.

There are a few different versions of paraphrase nominalism. Perhaps the most famous is *if-thenism*, which says that, for instance, '3 is prime' is best interpreted as expressing a conditional claim, such as 'If the entire series of natural numbers had existed, then it would have been the case that 3 was prime,' or 'Necessarily, if the entire series of natural numbers exists, then 3 is prime.' Versions of if-thenism have been developed by Putnam (1967, 1983), Horgan

[3] My own view is that the epistemological argument *isn't* good. See Footnote 8 for more on this.
[4] On some ways of categorizing views, certain kinds of *agnostics* (about the existence of mathematical objects) could count as anti-realists. But I won't worry about this here.

(1984), Hellman (1989), Dorr (2008), and Yablo (2017); moreover, a precursor to this view was endorsed by the early Hilbert (see Hilbert 1959 and his letters to Frege in Frege 1980). Finally, other versions of paraphrase nominalism have been endorsed by Wittgenstein (1956), Chihara (1990), Hofweber (2005), Rayo (2008, 2013), and Moltmann (2013).

The central idea behind most versions of paraphrase nominalism is that the non-face-value reading of mathematical sentences enables us to say that these sentences are true without committing to the existence of mathematical objects. For instance, if '3 is prime' really expresses a conditional claim, as if-thenists claim – in particular, if '3 is prime' really says that if the natural numbers had existed then it would have been the case that 3 was prime – then that sentence is true regardless of whether there's any such thing as the number 3.

2.3.2 Mathematical Error Theory and the Rejection of Premise [3]

Mathematical error theory (or, for short, just *error theory*) is the view that (a) the platonistic interpretation of mathematical discourse is correct – that is, our mathematical sentences and theories do *purport* to be about abstract mathematical objects, as platonists suggest – but (b) there are no such things as abstract objects, and so (c) our mathematical theories are not true. Thus, the idea here is that sentences like '3 is prime' are false, or untrue, for the same reason that, for instance, 'The Easter Bunny has big ears' is false or untrue – because just as there's no such thing as the Easter Bunny, so too there's no such thing as the number 3.[5] Error-theoretic views have been defended by Field (1980, 1989), me (Balaguer 1996, 1998, 2009), and Leng (2010).[6]

You might think error theory is untenable because you might think we have good reasons to think that our mathematical theories are true. But as we'll see in Section 3, it's harder than you might think to argue for the claim that our mathematical theories are strictly and literally true.

2.3.3 Deflationary-Truth Nominalism and the Rejection of Premise [2]

Deflationary-truth nominalism is the view that (a) ordinary mathematical sentences like '3 is prime' should be read at face value (i.e., as being of the form '*Fa*') and as making claims about mathematical objects; and (b) there are no such things as mathematical objects; but (c) our mathematical sentences are

[5] You might think, with Strawson (1950), that if there's no such thing as 3, then '3 is prime' is *neither true nor false*; I prefer the view that '3 is prime' is false in this scenario, and I'll assume that error theorists endorse that view, but nothing important turns on this.

[6] Also, related views have been defended by Melia (2000), Rosen (2001), and Yablo (2002a, 2002b, 2005).

still true. Views of this kind have been endorsed by Azzouni (2004, 2010) and Bueno (2005, 2009).

Deflationary-truth nominalism might seem hard to grasp. You might wonder how a sentence of the form '*Fa*' could be true if the singular term '*a*' doesn't refer to anything. How, for instance, could it be right to say that 3 is prime – that the number 3 has the property of being prime – if there's literally no such thing as the number 3? Isn't that like saying that (a) Mars is red, and (b) Mars doesn't exist? Isn't this incoherent?

But, in fact, there's an easy way to make coherent sense of deflationary-truth nominalism. The key is to understand the view as an empirical hypothesis about the meaning of the ordinary-language word 'true' – or, if you'd rather, about the ordinary concept of truth. When deflationary-truth nominalists say that, for instance, '3 is prime' could be true even if there were no such thing as the number 3, they're making a claim about the ordinary concept of truth – that is, about the concept that's expressed by the ordinary-language word 'true'. More specifically, they're saying that that concept applies in certain situations that most of us – mathematical platonists and error theorists and just about everyone else – think it *doesn't* apply in. So while you might *disagree* with deflationary-truth nominalism, the view is not incoherent.

Before moving on, it's worth distinguishing deflationary-truth nominalism from *Meinongianism*. The two views are superficially similar because Meinongians also claim that '3 is prime' could be true even if 3 didn't exist. But the similarity is only superficial because Meinongianism is a *realist* view, not an anti-realist view. Meinongians think that (a) there *is* such a thing as the number 3, and (b) the sentence '3 is prime' provides a true description of the number 3, but (c) despite all of this, 3 doesn't *exist*. Claims (a) and (b) already commit Meinongians to realism. They just have a nonstandard view of what's required for an object – a *real* object, an object that *is* – to count as "existing." In contrast to this, deflationary-truth nominalists wouldn't say that there *is* such a thing as the number 3; on the contrary, they think that there's no such thing at all.[7]

2.4 Some Arguments against Paraphrase Nominalism and Deflationary-Truth Nominalism

In Section 3, I'm going to respond to some objections to error theory, and in the process, I'll be trying to show that however implausible error theory might seem at first blush, it turns out to be fairly plausible, and indeed, there are reasons to

[7] Meinongian views have been endorsed by Routley (1980), Priest (2003, 2005), and of course Meinong (1904).

think it's the best version of anti-realism. But before I get to that, I want to argue against paraphrase nominalism and deflationary-truth nominalism.

2.4.1 An Argument against Paraphrase Nominalism

One problem with paraphrase nominalist views is that they commit to empirical claims about the meanings of ordinary mathematical utterances that are extremely implausible. For instance, if-thenism implies that ordinary utterances of '3 is prime' make conditional claims – for example, that *if the natural numbers had existed, then 3 would have been prime*. But this just seems to get wrong what ordinary people and ordinary mathematicians actually mean when they utter sentences like this. Certain kinds of *philosophers* – namely, those who don't believe in abstract objects – might think it would be *nice* if '3 is prime' really meant that if the natural numbers had existed, then 3 would have been prime. For we could then say that these sentences are true without committing to the existence of abstract objects. But in point of actual fact, it seems that, in ordinary English, '3 is prime' just *doesn't* mean that if the natural numbers had existed, then 3 would have been prime. That's just not what ordinary speakers mean when they utter '3 is prime,' and that's not what this sentence means in ordinary English.

More generally, we seem to have good reason to interpret ordinary mathematical sentences at face value; in other words, we have good reason to accept premise [1] and, hence, to reject all versions of paraphrase nominalism. The argument I have in mind here is based on the following, very plausible, interpretive principle:

> [IP] In general, when we're interpreting people's utterances, we should interpret them at face value unless there's evidence that they have positive intentions to be speaking nonliterally.

So if Jane utters the sentence 'Brindisi is in the heel of Italy's boot,' it's very likely that she intends (at some level, perhaps unconsciously) to be speaking nonliterally. And so it's acceptable to interpret Jane as speaking nonliterally in this case. But it seems that when ordinary people and ordinary mathematicians utter sentences like '3 is prime,' they usually don't intend (consciously or unconsciously) to be speaking nonliterally. For example, they don't intend to be expressing conditional propositions. On the contrary, it seems that they intend to be speaking literally. And so, given how plausible [IP] is as an interpretive principle, it seems that we should interpret ordinary speakers of mathematical sentences as speaking literally. And if this is right, then paraphrase nominalism is false.