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Compact Matrix Quantum Groups and Their Combinatorics

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Preface

The term 'quantum group' carries an air of mystery and physics, making it both frightening and fascinating. It is a difficult task to give an elementary explanation of what a quantum group is. In fact, the purpose of the first chapter of this book is to motivate and introduce as clearly as possible that notion. Nevertheless, the reader deserves a few explanations before trying to read it, and we will therefore try to give some elements of context concerning the theory of quantum groups.

The notion of quantum group is not an easy one to grasp. One of the main reasons for that is that there is, in a sense, no definition of 'quantum group' in general but rather a number of different families of objects which are given such a name. Another reason is that the definition of all these families of objects require tools beyond standard undergraduate mathematics. We will nevertheless try to give some intuition of what is going on, focusing, since this is our setting, on the compact case.

Let us start with a compact abelian group G and consider its *Pontryagin dual*, that is to say, the set $\widehat{G}$ of continuous group homomorphisms from G to the group $\mathbb{T}$ of complex numbers of modulus one. It is easy to see that $\widehat{G}$ is again an abelian group for pointwise multiplication. Moreover, the compact-open topology makes it a discrete topological space. Conversely, if we start with a discrete abelian group, then the Pontryagin dual will be a compact abelian group. The *Pontryagin duality theorem* then states that $\widehat{\widehat{G}} = G$.

To see why Pontryagin duality is interesting, we will look at the fundamental example $G = \mathbb{T}$. In that case, any group homomorphism $\varphi\colon \mathbb{T} \to \mathbb{T}$ is of the form $\varphi_n\colon z \mapsto z^n$ for some $n \in \mathbf{Z}$, so that $\widehat{\mathbb{T}} = \mathbf{Z}$. Consider now a finitely supported sequence $(a_n)_{n\in\mathbf{Z}}$. Seeing it as a function $a\colon \mathbf{Z} \to \mathbf{C}$, we can change it into a function $\widehat{a}\colon \mathbb{T} \to \mathbf{C}$ through the formula

$$\widehat{a}\colon z \mapsto \sum_{n\in\mathbf{Z}} a(n)\varphi_n(z) = \sum_{n\in\mathbf{Z}} a_n z^n.$$

In other words, finitely supported sequences and trigonometric polynomials can be put in one-to-one correspondence by Pontryagin duality. It is then natural to investigate what happens if we try to extend that correspondence to spaces of sequences in which the finitely supported ones are dense (for instance, square summable or continuous), and this is the field of Fourier analysis.

The upshot of the previous short discussion is that Pontryagin duality on $\mathbb{T}$ is an abstract, group-theoretic version of the theory of Fourier series. Thus, Pontryagin duality provides in full generality an analogue of the Fourier transform for any compact (or even locally compact, but this is another story) abelian group. It is then tempting to ask whether something similar is doable for non-abelian groups. Unfortunately, any group homomorphism $G \to \mathbb{T}$ must vanish on commutators since $\mathbb{T}$ is abelian, and so one has to find a different approach.

The theory of compact matrix quantum groups was initiated by S. L. Woronowicz during the 1980s [75], under the name *compact matrix pseudogroups*, to provide an approach to Pontryagin duality which works for arbitrary compact groups. Not only was it successful in doing so, but it also gave rise to many examples which do not relate to any compact or discrete group, the prominent ones being obtained by 'deforming' (see Section C.2 for an illustration) classical Lie groups like $SU(N)$. This was the birth of the theory.

Around the same time, V. Drinfeld coined the term *quantum group* in [35] to denote constructions of algebraic entities called *Hopf algebras* (see Appendix C for more on that notion), generalising the deformations of Lie algebras introduced by M. Jimbo in [44]. The correspondence between Lie groups and Lie algebras naturally suggests that there should be strong connections between the algebraic approach involving, for instance, the Lie algebra $\mathfrak{su}_N$ and the deformations of $SU(N)$ constructed by S. L. Woronowicz. And indeed, it was clear by then that there was a kind of duality relationship between both constructions, so that the terminology evolved in the work of S. L. Woronowicz to *compact quantum groups*. But despite their similarities, the two theories grew in different directions over the years so that they nowadays form distinct research topics.

The algebraic theory expanded quickly, partly because there was no strict definition of a quantum group in that setting, and so the term was applied to more and more classes of Hopf algebras with interesting properties, often connected to problems in mathematical physics. In the analytical theory, there was a precise definition to satisfy, in which the new algebraic examples did not necessarily fit. Consequently, the search for examples was slower. One important step (in particular, as far as this text is concerned) was the definition by S. Wang in [70] and [72] of the *universal compact quantum groups* which are nowadays denoted by S_N^+, O_N^+ and U_N^+. These objects, about which many things remain mysterious, have been a subject of intense study for the last twenty years and have led to important advances in the field, as we will see. Their crucial feature is that they are not deformations of classical groups, but rather 'liberations' obtained by removing the commutation relations in a suitable presentation of a classical group algebra. (This idea will be illustrated in Chapter 7.)

The first step in the study of a compact quantum group is the computation of its representation theory. For the previously mentioned examples of S. Wang, the representation theory was computed by T. Banica in a series of seminal works [3], [4] and [6]. These works revealed important combinatorial structures that proved crucial in the proof of many properties of the representation theory. The nice feature of these combinatorial structures is that they are based on *diagrams* in the sense that one represents operators by some drawings and can do computations directly on the drawings by simply using intuitive pictorial properties. After a period of maturation, the connection between these combinatorial structures and compact quantum groups was formalised by T. Banica and R. Speicher in [18] in a way which is both natural and quite elementary. The fundamental idea is to express the relations defining the algebras of functions on the quantum groups in terms of intertwiners, themselves defined from partitions of finite sets, which naturally give back the diagrams. Although the difference may seem small, partitions offer great flexibility for computations and suggest the use of many results from combinatorics and free probability theory. This change of paradigm was a milestone in the study of compact quantum groups and has had many important consequences. Moreover, it continues to lead to unexpected connections with other fields of mathematics like quantum information theory (see, for instance, the beginning of Chapter 1, as well as Chapter 8).

At the date of this writing, the study of compact quantum groups defined through the combinatorics of partitions is a very active topic with a well-established theoretical basis. However, even though there exist books on the general theory of compact quantum groups, there is none detailing the combinatorial aspects and their connections to other subjects. The purpose of this book is therefore to give a comprehensive introduction to compact matrix quantum groups as well as to the combinatorial theory of partition quantum groups. It has been written and designed for students and should be particularly fit for a one-semester graduate course as well as for self-study by a motivated graduate (or even a highly motivated undergraduate) student. It is nevertheless our hope that it can also be of use to more advanced mathematicians who want to learn the subject or have a glimpse of its recent outcomes, but are afraid to get lost in the growing scientific literature.

Before outlining the contents of the text, let us comment on its differences with other books on the subject. As already mentioned, the theory of compact quantum groups was initially developed in the setting of operator algebras, and that theory is detailed in [60] and in [69]. It is known, however, that there is an equivalent algebraic approach to the subject through specific Hopf algebras called *CQG-algebras* (see Appendix C), which is explained in detail in [69]. These introductory books share the same caveat: the reader is supposed to have learned some prerequisites, operator algebras or Hopf algebras, before starting to study the main subject. Because we wanted to teach a course on compact quantum groups in a university where no course on operator algebras or Hopf algebras was available, we had to find another way.

It turns out that by focusing on compact *matrix* quantum groups (i.e. those having a faithful finite-dimensional representation), one can prove all the important theorems without resorting to any result from the theory of operator algebras or Hopf algebras. As a consequence, we are able to provide here a text which is completely self-contained for anyone having completed undegraduate studies in mathematics in any university. More precisely, the prerequisites for reading the book are only basic algebra, possibly (but not necessarily) including some elementary notions concerning the representation theory of finite-dimensional complex algebras. To make the book as self-contained as possible, we have included, where necessary, a short treatment of some more advanced algebraic notions like universal algebras or tensor products as well as an appendix containing statements and proofs of two results from representation theory (Burnside's theorem and the double commutant theorem) which are needed in the course of the book.

Let us now briefly outline the contents and organisation of the text. An important point is that since it was designed for a course, it is intended to be read rather linearly, at least as far as the first two parts are concerned. For clarity, it is divided into three main parts, themselves split into chapters.

- Part I contains the basic theory of compact matrix quantum groups, going from the very definition of these objects in Chapter 1 to the complete description of their representation theory in Chapter 2. The definition is motivated by the example of quantum permutations and their connection with quantum information theory through the *graph isomorphism game*. Along the way, we briefly introduce universal algebras and tensor products (over the complex numbers) in case the reader is not familiar with these notions.

- Part II introduces the connection between partitions of finite sets and quantum groups through the notion of *partition quantum groups*. Chapter 3 introduces all the necessary material as well as a proof of the celebrated Tannaka–Krein–Woronowicz theorem, which is the cornerstone of the definition of partition quantum groups. In Chapter 4 we give a detailed treatment of the representation theory of partition quantum groups, which is then applied to the basic examples S_N^+ and O_N^+. We conclude this part with the construction of the analogue of the Haar measure. We take this occasion to make the bridge with the analytical approach to quantum groups, though without full proofs. We also give some applications to non-commutative probability theory.
- Part III contains extra topics of two different types. Chapters 6 and 7 introduce the coloured versions of partition quantum groups which enable one to deal with the unitary quantum groups U_N^+ and H_N^{s+}. This is also an occasion for discussing side topics like free complexification, free wreath products and quantum automorphism groups of graphs, which are not available in any book so far. Chapter 8, on the other hand, treats the link between quantum permutation groups and quantum information theory. This motivates questions of residual finite-dimensionality which are detailed in the final section. This shows the power of the combinatorial methods developed in the text.
- At the end of the book there are three appendices. Appendix A gives the statements and proofs of two elementary results on complex representations of matrix algebras which are needed in the text. Then, Appendix B shows how one can recover the usual description of the representation theory of classical compact matrix groups out of what has been done in the quantum setting. Finally, Appendix C is an invitation to general compact quantum groups, where we indicate how one can build a larger theory from what has been done in the main text and hint at some subtle issues that then arise.

The material for this book comes from two sources. The first one is a mini-course taught in the masterclass ‘Subfactors and Quantum Groups’ at Copenhagen in May 2019. The second is a graduate course entitled ‘Introduction to Compact Matrix Quantum Groups and Their Combinatorics’ taught at University Paris-Saclay in 2020 and 2021. It is a pleasure to thank both institutions, and the organisers of the masterclass, Rubén Martos and Ryszard Nest, for giving me opportunities to teach this subject. I am also deeply indebted to all those who have journeyed with me in the fascinating world of compact quantum groups, and in particular to Etienne Blanchard, Roland Vergnioux, Pierre Fima, Julien Bichon, Adam Skalski and Moritz Weber.