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Algebraic Combinatorics and the Monster Group

Edited by

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Preface

The content of this collection is best viewed via the relationship and the interaction between examples, axiomatization, and theory. In [1] Simon Norton wrote: ‘It is often said that Euclid’s “Elements” was intended not so much as an introduction to geometry but to show how to construct the five Platonic Solids; these solids had (in modern terminology) the most elaborate symmetry groups that had been encountered at that time, and were known to be of special interest. On the way to constructing them it would have been necessary to expound most of the geometry that was known at the time.’

The wideness of this viewpoint might be arguable, but we clearly observe a similar situation in recent mathematics. Ernst Witt constructed his designs in [2] to remove forever any doubts about the existence of the Mathieu groups M_{11}, M_{12}, M_{22}, M_{23}, and M_{24} discovered by Émile Mathieu some 70 years earlier [3]. The future development of the theory of designs demonstrated that these are the pearls of design theory. The largest of the Witt designs is on 24 points and it is the unique S(5,8,24).

In the 1960s, when the Leech lattice came to John Conway’s attention, it became clear that the ‘purpose’ of the Witt design $S(5,8,24)$ associated with the largest Mathieu group M_{24} was to serve through the Golay code as the frame of the Leech lattice. The fundamental monograph [4] of 1 318 grams contains many lattices and their theories, but essentially it is to justify that the Leech lattice is the most special one. The most common characterization of the Leech lattice is as the unique 24-dimensional, even, unimodular lattice without roots. If the roots are allowed, we obtain in addition 23 Niemeier lattices, whose existence can be justified by 23 constructions of the Leech lattice, one from each Niemeier lattice.

In its turn the ‘purpose’ of the Leech lattice Λ and its automorphism group $Co_0 = 2.Co_1$ is the structure of the centralizer $C \cong 2_+^{1+24}.Co_1$ of an involution in the Monster group, where $O_2(C)/Z(C)$ is isomorphic to $\Lambda/2\Lambda$ as a module for $C/O_2(C) \cong Co_1$. However, originally the way to the Monster was opened through a different door.

In the late 1960s, Bernd Fischer [5] axiomatized the order product property of the transpositions of the symmetric groups to build up a theory of 3-transposition groups. The highest point of this theory was discovery of three sporadic 3-transposition groups Fi_{22}, Fi_{23}, and Fi_{24}. The former two and the index 2 commutator subgroup Fi'_{24} of the last one are now known as Fischer's sporadic simple groups. The maximal set of pairwise commuting transpositions in Fi_{24} contains precisely 24 transpositions and carries a structure of Witt design $S(5,8,24)$ with the Mathieu group M_{24} being the action induced on this set by its stabilizer (which is a non-split extension of the dual Golay code module 2^{12} by M_{24}). The 'purpose' of the largest Fischer's sporadic 3-transposition group Fi'_{24} extended by its Schur multiplier of order 3 is to serve as the centralizer of an order 3 subgroup in the Monster group. But this is still not the original path to the Monster.

The non-sporadic 3-transposition groups include besides the symmetric groups the classical symplectic, orthogonal, and unitary groups. Fischer relaxed the axioms by admitting products of order 4, which allowed in the exceptional Lie-type groups including ${}^2E_6(2){:}\,2$. Further on, a $\{3,4\}$-transposition group, having $2\cdot{}^2E_6(2){:}\,2$ as the transposition centralizer, turned out to be another sporadic simple group now known as Fischer's Baby Monster and is denoted by BM.

The non-split central extension $2\cdot BM$ of the Baby Monster is an example of a 6-transposition group (in which products of order 5 are absent) and it served as the centralizer of an involution in another 6-transposition group, which is the Monster group M. Bernd Fischer came to this evidence in 1973 and, independently, in the same year the existence of the Monster was suggested by Robert Griess.

In the 1970s, there was an intensive study of various properties of the then-hypothetical Monster M and some of the results were left unpublished. It was shown that the minimal faithful complex representation of M has degree at least 196 883; Simon Norton showed that if such a representation exists, then it carries a non-zero inner product and a non-associative algebra product, which are unique up to rescaling. John Thompson proved the uniqueness of M subject to the structure of involution centralizers and the existence of a 196 883 representation [6]. In 1980, Griess constructed the 196 883-dimensional algebra, and the Monster itself [7]. The construction was improved by John Conway [8] who in particular added the identity to the algebra, increasing its dimension by 1. The extended algebra is called Conway–Griess–Norton algebra or simply the Monster algebra.

The theory around the Monster was initiated by the observation made by John McKay that 196 883 + 1 = 196 884, where the right-hand side is the linear coefficient of the most celebrated modular form $J(q)$. A remarkable sequence of events, which goes under the name of Monstrous Moonshine [9], culminated (but was not concluded) in the construction of the Vertex Operator Algebra (VOA) $(V^\natural, *_n)$ known as the Moonshine Module [10]. This is an infinite-dimensional integer-graded algebra with infinitely many products $*_n$ satisfying infinitely many Jordan-type relations. In this context the Monster algebra and the inner product are realized as on grade 2 operators: $(V_2, *_3, *_1)$.

The axiomatic for VOAs was designed by Richard Borcherds [11] with a close look at the Moonshine Module construction [10]. Thus the theory was designed for the construction of $V^\natural$ based on the Leech lattice, although it can also be applied to any Niemeier lattice, and even in two different ways: twisted and untwisted constructions. There are quite remarkable isomorphisms between the lattice VOAs, so in total one obtains 39 (rather than 48) lattice VOAs with central charge 24 (which inherits the dimension of the lattice). These VOAs are rational, meaning that their theta series are the modular invariant $J(q)$ with the constant term (properly normalized) being the dimension of the level one sector V_1. Within this class, the Moonshine module (twisted construction based on the Leech lattice) is characterized by the condition $V_1 = 0$ (which has a deep analogy with the rootless property of the Leech lattice).

The main features of the VOA theory are now understood to be implicit in the physical string theory, and in physics VOAs correspond to the 2-dimensional meromorphic conformal field theories. In 1993, A. N. Shellekens [12] gave physical evidence for the existence of exactly 71 strongly regular holomorphic VOAs with central charge 24, including the lattice constructions. Since then the attempts to obtain a rigorous mathematical proof of this evidence, including constructions of the predicted VOAs, became one of the most fundamental problems in the theory of VOA.

Thus, through moonshine VOA, the Monster was placed in the centre of a remarkable theory with deep connections with modern physics through infinite-dimensional algebras. The attempts to axiomatize the finite-dimensional Monster algebra itself led to success when Alexander Ivanov axiomatized in [13] certain properties of the $2A$-axes of the Monster algebra from [8] under the name of Majorana axes. This led to Majorana theory, which enables identification of the isomorphism type of various subgroups in the Monster generated by a set of $2A$-involutions.

In [1] Simon Norton wrote ‘one may consider an “ideal” ATLAS whose culmination was a simple explanation of the existence of the Monster, with properties of many smaller groups being covered on the way’. We hope that in its completion, Majorana theory might constitute the material for such an ideal ATLAS, where the Monster is characterized as the automorphism group of the largest Majorana algebra, which is the Monster algebra, and all $2A$-generated subgroups of the Monster classified through the subalgebras generated by the relevant set of $2A$-axes.

In [14] Jonathan Hall, Felix Rehren, and Sergey Shpectorov have relaxed certain Majorana axioms to obtain a class of axial algebras, which contains more examples including Jordan and Matsuo algebras, and exhibits a richer theory allowing the universal algebra construction and more.

We can now review the papers in the first part of this collection. The chapter by Atsushi Matsuo provides an introduction to the theory of Vertex Algebras and Vertex Operator Algebras. It illustrates, in much details, the most important examples of the algebras, including free bosons and lattice constructions. The survey both enables a newcomer to join the subject and provides an outsider the logic and starting content of the subject.

The chapter by Hiroshi Yamauchi describes the role of the theory and examples of 3-transposition groups in vertex operator algebras. These groups particularly appear from Matsuo algebras, which can be viewed as Majorana algebras missing the $\frac{1}{4}$-eigenspaces of Majorana axes. An important feature is that 3-transposition groups appear in a wider context of VOAs than the particular moonshine module associated with the Monster.

Ching Hung Lam is one of the leading and most active players in the almost 30-year-long attack on the Schellekens conjecture. In his chapter, he gives an account of the present state of proving this conjecture, which is very close to completion. All the 70 anticipated analogues of the moonshine module have been constructed by now. The technique of these constructions based on orbit-folding is clearly explained.

The chapter by Ulrich Meierfrankenfeld and Sergey Shpectorov is an almost classical and ingenious work on maximal 2-local subgroups in the Monster and Baby Monster and gives a unique insight into the local structure of these groups. Although written at the turn of the millennium and highly circulated, it was previously unpublished and we are happy to present it in this volume.

The chapter by Alexander A. Ivanov gives the account of the current status of Majorana theory. The Majorana representations of small and not so small groups are now classified, the main achievement being the classification of the saturate Majorana representation of A_{12}. At least two working computer packages for calculating Majorana representations are now available. This suggests the current strategy comprising classifying small sub-representations by a computer and assembling them together by hand.

The machinery for developing Majorana theory and axial algebras lies in the scope of Algebraic Combinatorics to which the second part of this collection is devoted. In particular, the central combinatorial object for both the theories is the association scheme of the Monster group acting by conjugation on the class of its $2A$-involutions. It is precisely through this scheme that the methods of algebraic combinatorics were originally used by Simon Norton to establish the uniqueness of the Monster. The procedure of recovering the Monster algebra from the structure of the centralizers of various elements encoded in the structure constants of the association scheme brought the term 'Norton algebras' into algebraic combinatorics [15]. We discuss some aspects of algebraic combinatorics also through relationships between examples, axiomatics, and theory illustrated by the chapters in Part II.

The understanding of the finite simple groups of Lie type started in the nineteenth century with linear groups and moved on to the classical groups, symplectic, orthogonal, and unitary, summarized in [16]. In the first half of the twentieth century exceptional groups of types G_2, E_6, and F_4 were constructed by A. A. Albert as the automorphism groups of exceptional Jordan algebras (now reincarnated in the form of axial algebras). At the beginning of the 1960s, these and other exceptional groups were axiomatized by Jacques Tits [17], [18] under the names of buildings and BN-pairs. The leading chapter of Part II, that by Hendrik Van Maldeghem, is a brilliant survey of some recent developments in the theory of buildings, geometries, and varieties related to the exceptional groups. Various exceptional geometries can be arranged in Freudenthal–Tits Magic Squares so that different series can be read off in both the rows and the columns. This square and its remarkable properties is at the centre of the chapter.

When geometries are axiomatized as buildings or even as point–line systems, they are no longer subspaces in any vector space but just abstract sets with incidence relations satisfying certain axioms. It appears fruitful to introduce the vector space structure by considering representations of geometries. A survey of this approach is given in the chapter by Ilaria Cardinali, Luca Giuzzi, and Antonio Pasini.

Given a point–line incidence system, one can define subspaces and hyperplanes, which are crucial in understanding representations. An ovoid is a subspace and the question of when it is maximal is discussed in the chapter by Antonio Pasini and Hendrik Van Maldeghem.

The representation theory is a very important tool in Majorana theory through algebraic combinatorics. So far, mainly ordinary representations appear, although modular ones are expected to come through modular Moonshine or otherwise. We are pleased to introduce a chapter in the collection written by leading experts in representation theory, Tatsiana S. Busel and Irina D. Suprunenko. They discuss rather delicate features of highest weight representations of algebraic groups. With sadness we report that Irina Dmitrievna Suprunrnko passed away on 10 August 2022.

The axiomatics of distance-regular graphs were introduced by Norman Biggs in the late 1960s [19]. By that time the classical generalized polygons were known, as were some sporadic examples including the Livingstone graph for the sporadic group J_1 of Janko. The theory of distance-regular graphs was at the centre of algebraic combinatorics in the 1970s and 1980s, which led to the celebrated monograph [20]. Jack H. Koolen is a leading researcher in the modern theory of distance-regular graphs. The chapter he wrote jointly with Jongyook Park and Qianqian Yang in the collection is on distance-regular graphs with classical parameters.

In the terminology of [15] the distance-regular graphs are precisely the *P*-polynomial association schemes. If the scheme is also *Q*-polynomial, then the usual matrix and Hadamard products are tri-diagonalizable in suitable bases. This property was axiomatized by Paul Terwilliger under the name of Terwilliger algebras. These algebras were studied by Paul together with Tatsuro Ito, and a survey on these algebras is in their chapter in the collection.

The theory of non-symmetric (analogues of) association schemes is less developed. A nice result in the chapter by Takuya Ikuta and Akihiro Munemasa contributes to this theory.

These days computer programs designed for calculations in algebraic combinatorics are often included in packages of the GAP system [21]. The GAP packages – GRAPE for graph theory, DESIGN for design theory, and FinInG for finite incidence geometry – described by Leonard Soicher in the concluding chapter, are of crucial importance.

The authors of the collection belong to a dynamic mathematical community. The editor has organized a number of meetings for this community starting with the 1991 conference in Vladimir, Russia (jointly with Igor Faradjev and Mikhail Klin). The latest was a hybrid 2021 conference in Rogla, Slovenia (jointly with Elena Konstantinova). Below is a photo from a London 2013 conference that shows many contributors to this collection.

The Editor

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Edited by Alexander A. Ivanov
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