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Part I

The Monster

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Lectures on Vertex Algebras

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Abstract

The purpose of the present chapter is to explain the basics of vertex algebras, as well as some more advanced topics on vertex operator algebras, to the reader mainly in the fields of group theory and algebraic combinatorics.

CONTENTS

Introduction

The Monster, the largest sporadic finite simple group of order

$$2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$$
$$= \underbrace{808017424794512875886459904961710757005754368000000000}_{54 \text{ digits}},$$

is known to be realized as the automorphism group of the moonshine module $\mathbf{V}^{\natural}$, a distinguished example of a vertex operator algebra, equipped with a grading of the shape

$$\begin{array}{ccccccccc} \mathbf{V}^\natural & = & \mathbb{C}\mathbf{1} & \oplus & 0 & \oplus & \mathbf{B}^\natural & \oplus \mathbf{V}_3^\natural \oplus \mathbf{V}_4^\natural \oplus \cdots , \\ \dim & & 1 & & 0 & & 196884 & \end{array}$$

of which the dimensions of the homogeneous subspaces satisfy

$$q^{-1}\sum_{n=0}^{\infty} \dim \mathbf{V}_n^\natural q^n = j(\tau) - 744 \\ = q^{-1} + 0 + 196884q + 21493760q^2 + \cdots , \tag{1}$$

where $j(\tau)$ is the elliptic modular function and $q = e^{2\pi\sqrt{-1}\tau}$.

The 196884-dimensional subspace $\mathbf{B}^\natural$ of degree 2 inherits a structure of a commutative nonassociative algebra with unity equipped with a nondegenerate symmetric invariant bilinear form, which we call the *Griess–Conway algebra*, as suggested by S. P. Norton. The algebra $\mathbf{B}^\natural$ is a variant of the algebras constructed by R. L. Griess in [61] to prove the existence of the Monster, and it is indeed the same as the algebra constructed by J. H. Conway in [38].

The notion of vertex algebras was introduced by R. E. Borcherds in the seminal paper [32] in 1986 by axiomatizing properties of infinite sequences of operators constructed from even lattices that generalize those considered for the root lattices of ADE type in the famous *Frenkel–Kac construction,* achieved by I. B. Frenkel and V. G. Kac in [57], to realize representations of affine Kac–Moody algebras associated with simple Lie algebras of the corresponding type. Such sequences of operators are related to the *vertex operators* in string theory, whence the term *vertex algebra*. The vertex operator is actually not a single operator but an infinite series with operator coefficients. The concept of vertex algebras can be seen to be a mathematical formulation of what is called the *operator product algebra* or the *chiral algebra* in physics.

Borcherds then applied vertex algebras to the study of the Monster via the moonshine module $\mathbf{V}^\natural$, which was previously introduced by I. B. Frenkel, J. Lepowsky, and A. Meurman [59] as a vector space equipped with some structures, and achieved in [33], with numerous outstanding ideas and works, the proof of the Conway–Norton conjecture, the conjecture that states the famous moonshine phenomena relating representations of the Monster and certain modular functions, the simplest among which is (1).

The concepts of vertex operator algebras (VOA) and their modules, in turn, were formulated by I. B. Frenkel, J. Lepowsky, and A. Meurman in [1] in order to set up appropriate "algebras" and "modules" by modifying those for vertex algebras. More precisely, a VOA is not just a vertex algebra, but a pair consisting of a vertex algebra and its element generating a representation of the Virasoro algebra satisfying a number of conditions that would make it suitable for applications.

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Table 1 Codes, lattices and VOAs

Doubly even codes	Postive-definite even lattices	VOAs
Length	Rank	Central charge
Weight enumerator	Theta function	Conformal character
Self-dual	Unimodular	Holomorphic
Extended Hamming code H_8	Gosset lattice E_8	Lattice VOA $\mathbf{V}_{E_8}$
Extended Golay code G_{24}	Leech lattice Λ	Moonshine module $\mathbf{V}^\natural$
Mathieu group M_{24}	Conway group C_0	Monster $M = F_1$

For example, VOAs are assumed to be graded by integers with the homogeneous subspaces being finite-dimensional, so that one may consider the conformal character, the generating series of dimensions such as (1).

In fact, important applications of vertex algebras are often based on the properties of the Virasoro algebra, thus justifying the definition of VOAs.

The moonshine module $\mathbf{V}^\natural$ indeed carries a natural structure of a VOA. It possesses a distinguished position among VOAs when viewed through the famous analogies of binary codes, lattices, and VOAs as indicated in Table 1, although the uniqueness of $\mathbf{V}^\natural$ conjectured in [1], which is an analogue of the uniqueness of the extended Golay code G_{24} and the Leech lattice Λ, is yet to be settled. Thus the concept of VOAs is as natural as those of binary codes and lattices. However, even constructing a single example of a VOA is not so easy.

In Section 1.1, we will describe the definition of vertex algebras after preliminary sections, and then proceed to realization of vertex algebras by formal series with operator coefficients in Section 1.2, where the concept of modules over vertex algebras will also be introduced. Such realization enables us to state and prove the existence of vertex algebra structures under certain circumstances. Standard examples of vertex algebras will be described in Section 1.3.

Section 1.4 is devoted to construction of the vertex algebras associated with even lattices, where commutation relations of vertex operators play fundamental roles. In Section 1.5, we will explain the definition and construction of what are called *twisted modules* over vertex algebras by repeating the arguments of the previous sections in slightly more general settings, which enables one to construct the moonshine module $\mathbf{V}^\natural$ as a module over a fixed-point subalgebra of the Leech lattice vertex algebra by a lift of the (-1)-involution.

In Section 1.6, we will give brief accounts of theory of VOAs including fusion rules and modular invariance. We will then finish the sections by mentioning properties of the moonshine module and their variants that opened ways to new research directions.

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1.1 Axioms for Vertex Algebras

A vertex algebra is a vector space equipped with countably many binary operations indexed by integers satisfying a number of axioms.

In Section 1.1, we start with preliminary sections on algebras and formal series and then describe the definition of vertex algebras and some consequences of the axioms. We will give a few examples: the commutative vertex algebras, the Heisenberg vertex algebra, and a Virasoro vertex algebra as a vertex subalgebra of the Heisenberg vertex algebra.

We will work over a field $\mathbb{F}$ of any characteristic not 2, thus vector spaces and linear maps are always over such a field $\mathbb{F}$, unless otherwise stated. We denote the set of integers by $\mathbb{Z}$ and that of nonnegative integers by $\mathbb{N}$.

1.1.1 Preliminaries on Algebras

For a vector space $\mathbf{M}$, consider the set $\operatorname{End}\mathbf{M}$ of all operators (endomorphisms) acting on $\mathbf{M}$. The symbol $I = I_{\mathbf{M}}$ refers to the identity operator.

For an operator $A \in \operatorname{End}\mathbf{M}$, we will denote the value of A at $v \in \mathbf{M}$ by juxtaposition:

$$A\colon \mathbf{M} \longrightarrow \mathbf{M}, \quad v \mapsto Av.$$

Compositions of operators, also written by juxtaposition, are taken from right to left unless specified by parentheses: for $A, B, C \in \operatorname{End}\mathbf{M}$ and $v \in \mathbf{M}$,

$$ABC = A(BC), \quad ABCv = A(B(Cv)), \quad \text{etc.}$$

The commutator of operators is denoted by the bracket as

$$[A, B] = AB - BA$$

for $A, B \in \operatorname{End}\mathbf{M}$.

1.1.1.1 Associative Algebras

Let us first recall the definition of associative algebras. We will always assume that associative algebras are unital.

An *associative algebra* is a vector space $\mathbf{A}$ equipped with a bilinear map

$$\mathbf{A} \times \mathbf{A} \longrightarrow \mathbf{A}, \quad (a, b) \mapsto ab,$$

called *multiplication* or the *product operation*, satisfying the following axioms:

(A1) Associativity. For all $a, b, c \in \mathbf{A}$:

$$(ab)c = a(bc).$$

(A2) Unity. There exists an element $\mathbf{1} \in \mathbf{A}$ such that for all $a \in \mathbf{A}$:

$$\mathbf{1}a = a \text{ and } a\mathbf{1} = a.$$

The element $\mathbf{1} \in \mathbf{A}$ in (A2) is uniquely determined by the conditions therein and called the *unity* of $\mathbf{A}$,

For a vector space $\mathbf{M}$, the set $\operatorname{End}\mathbf{M}$ of all operators acting on $\mathbf{M}$ becomes an associative algebra by composition of operators, of which the unity is the identity operator.

1.1.1.2 Modules over Associative Algebras

A *module* over $\mathbf{A}$, or an $\mathbf{A}$-*module,* is a vector space $\mathbf{M}$ equipped with a bilinear map

$$\mathbf{A} \times \mathbf{M} \longrightarrow \mathbf{M}, \quad (a, v) \mapsto av,$$

called an *action* of $\mathbf{A}$ on $\mathbf{M}$, satisfying

(AM1) Associativity. For all $a, b \in \mathbf{A}$ and $v \in \mathbf{M}$:

$$(ab)v = a(bv).$$

(AM2) Identity. For all $v \in \mathbf{M}$: $\mathbf{1}v = v.$

For $a \in \mathbf{A}$, the operator on $\mathbf{M}$ sending v to av is called the *action* of a on $\mathbf{M}$.

For an $\mathbf{A}$-module $\mathbf{M}$, consider the map assigning the action on $\mathbf{M}$ to each element of $\mathbf{A}$:

$$\rho_{\mathbf{M}} : \mathbf{A} \longrightarrow \operatorname{End}\mathbf{M}, \quad a \mapsto [v \mapsto av].$$

Then this map is a homomorphism of algebras. Such a homomorphism is called a *representation* of $\mathbf{A}$ on $\mathbf{M}$. The concepts of modules over $\mathbf{A}$ and representations of $\mathbf{A}$ are essentially the same.

The algebra **A** itself becomes an **A**-module by the product operation, for which the *left action* of $a \in \mathbf{A}$ sending x to ax is called *left multiplication* by a. The corresponding representation

$$\rho_{\mathbf{A}} : \mathbf{A} \longrightarrow \operatorname{End} \mathbf{A}, \quad a \mapsto [x \mapsto ax]$$

is an isomorphism of algebras onto its image.

1.1.1.3 Lie Algebras

A *Lie algebra* is a vector space **L** equipped with a bilinear map

$$[\ ,\ ] : \mathbf{L} \times \mathbf{L} \longrightarrow \mathbf{L}, \quad (X, Y) \mapsto [X, Y],$$

called the *bracket operation*, satisfying

(1) For all $X, Y, Z \in \mathbf{L}$:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

(2) For all $X \in \mathbf{L}$:

$$[X, X] = 0.$$

As the base field is assumed to be not of characteristic 2, the set of the two conditions is equivalently replaced by

(L1) Jacobi identity. For all $X, Y, Z \in \mathbf{L}$:

$$[[X, Y], Z] = [X, [Y, Z]] - [Y, [X, Z]].$$

(L2) Antisymmetry. For all $X, Y \in \mathbf{L}$:

$$[X, Y] = -[Y, X].$$

Throughout the sections, we will take the latter conditions (L1) and (L2) as the axioms for Lie algebras and call the identity in (L1) the *Jacobi identity*, although this term usually refers to (1) rather than (L1).

For a vector space **M**, the space End **M** becomes a Lie algebra by the commutator of operators, for which the Jacobi identity

$$[[A, B], C] = [A, [B, C]] - [B, [A, C]], \quad A, B, C \in \operatorname{End} \mathbf{M}$$

trivially holds by cancellation of terms in

$$\begin{aligned}(ABC - BAC) &- (CAB - CBA) \\ &= \big((ABC - ACB) - (BCA - CBA)\big) \\ &\quad - \big((BAC - BCA) - (ACB - CAB)\big).\end{aligned}$$

A variant of this simple observation will serve as a basis for the Borcherds identity, the main identity for vertex algebras, where A, B, C are replaced by series with operator coefficients. (See Subsection 1.2.3.1.)

Similarly, any associative algebra $\mathbf{A}$ is regarded as a Lie algebra by the commutator

$$[a, b] = ab - ba, \quad a, b \in \mathbf{A}.$$

We will denote this Lie algebra by $\mathbf{L}(\mathbf{A})$.

Note 1.1. A vector space $\mathbf{L}$ equipped with a bracket operation satisfying (L1) but not necessarily (L2) is called a (left) *Leibniz algebra* and the property (L1) is called the (left) *Leibniz identity*. Note that (L1) is equivalently written as

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]],$$

which says that the operations of taking the brackets by elements of $\mathbf{L}$ are derivations with respect to the bracket operation itself.

1.1.1.4 Modules over Lie Algebras

An $\mathbf{L}$-*module*, or a *module* over $\mathbf{L}$, is a vector space $\mathbf{M}$ equipped with a bilinear map

$$\mathbf{L} \times \mathbf{M} \longrightarrow \mathbf{M}, \quad (X, v) \mapsto Xv,$$

satisfying

(LM) For all $X, Y \in \mathbf{L}$ and $v \in \mathbf{M}$:

$$[X, Y]v = X(Yv) - Y(Xv).$$

For an $\mathbf{L}$-module $\mathbf{M}$, consider the map assigning the corresponding action on $\mathbf{M}$ to each element of $\mathbf{L}$:

$$\rho_{\mathbf{M}} : \mathbf{L} \longrightarrow \operatorname{End} \mathbf{M}, \quad X \mapsto [v \mapsto Xv].$$

Then this map is a homomorphism of Lie algebras. Such a homomorphism is called a *representation* of $\mathbf{L}$ on $\mathbf{M}$. The concepts of modules over $\mathbf{L}$ and representations of $\mathbf{L}$ are essentially the same.

The Lie algebra $\mathbf{L}$ itself becomes an $\mathbf{L}$-module by the bracket operation, for which the action of $X \in \mathbf{L}$ sending Y to $[X, Y]$ is called the *adjoint action* of X, and the corresponding representation

$$\rho_{\mathbf{L}} : \mathbf{L} \longrightarrow \operatorname{End} \mathbf{L}, \quad X \mapsto [Y \mapsto [X, Y]],$$

is called the *adjoint representation* of $\mathbf{L}$.