INTRODUCING STRING DIAGRAMS

String diagrams are powerful graphical methods for reasoning in elementary category theory. Written in an informal expository style, this book provides a self-contained introduction to these diagrammatic techniques, ideal for graduate students and researchers. Much of the book is devoted to worked examples highlighting how best to use string diagrams to solve realistic problems in elementary category theory. A range of topics are explored from the perspective of string diagrams, including adjunctions, monad and comonads, Kleisli and Eilenberg–Moore categories, and endofunctor algebras and coalgebras. Careful attention is paid throughout to exploit the freedom of the graphical notation to draw diagrams that aid understanding and subsequent calculations. Each chapter contains plentiful exercises of varying levels of difficulty, suitable for self-study or for use by instructors.

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The Art of Category Theory

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Dedicated to Anja, Lisa, and Florian

Dedicated to Nuala, Florin, and Kiko

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Prologue

Why should you read this book? Proofs in elementary category theory typically involve either the pasting together of commuting diagrams (Mac Lane, 1998) or calculational reasoning using chains of equalities (Fokkinga and Meertens, 1994). The first style is sometimes referred to as "diagram chasing" since the focus of a proof is chased around a diagram; the second is also known as "squiggoling" since it often involves the use of "squiggly" symbols.

Both styles have their merits. Commutative diagrams capture an abundance of type information, and invoke a certain amount of visual intuition. Equational proofs are familiar from many other branches of science and mathematics. They are also compact and carry a clear orientation from assumptions to goals.

Unfortunately, both styles also have serious limitations. The usual equational style of reasoning forces us to abandon the vital type information that guides proof attempts and protects against errors. Commuting-diagram-style proofs retain the type information. Sadly, the proof style is unfamiliar to many in other fields of mathematics and computer science, and the resulting proofs often lack motivation and a clear direction of argument. Further, much of the effort in these proofs can be consumed by trivial administrative steps involving functoriality, handling naturality componentwise, and the introduction and elimination of identities.

In order to recover the best of both these approaches, in this monograph we advocate the use of *string diagrams* (Penrose, 1971), a two-dimensional form of notation, which retains the vital type information while permitting an equational style of reasoning. The book can be seen as a graphical approach to elementary category theory, or dually as a course in string diagrammatic reasoning with elementary category theory as the running example.

The diagrammatic notation silently deals with distracting bookkeeping steps, such as naturality and functoriality issues, leaving us to concentrate on the essentials. This is an important aspect in any choice of notation, as advocated by Backhouse (1989). The resulting diagrams and proofs are highly visual and we can often exploit topological intuition to identify suitable steps in our reasoning. We aim to illustrate the efficiency and elegance of string diagrams with the many explicit calculations throughout the book.

There is a lack of expository material on the application of string diagrams. Probably the most substantial exception appears in the draft book (Curien, 2008a), and the shorter related conference paper (Curien, 2008b). These works are complementary to the current text, taking a rather different direction in their exploration of string diagrams. The standard introductory texts generally omit string diagrams, with the exception of a brief mention by Leinster (2014). This is unfortunate as it makes these beautiful methods unduly hard to pick up, without either access to experts, or detailed knowledge of the academic literature. This monograph aims to address this gap.

This is *not* a book on the mathematical foundations of string diagrams. Instead, it is a guide for users, with many explicit calculations illustrating how to apply string diagrams. The aesthetics of string diagrams are important; a bad drawing can become a baffling mass of spaghetti, whereas good diagrammatic choices can render proof steps almost embarrassingly obvious. We explore how to exploit this artistic freedom effectively throughout the text.

Can you read this book? We have aimed for a minimum of prerequisites for readers, beyond a certain level of mathematical maturity. Ideally, you should have some familiarity with the basic objects of discrete maths, such as preorders and partial orders, monoids, and graphs, at the level of Gries and Schneider (2013), for example.

Readers who have already been introduced to category theory will find much that is new, as our consistent use of string diagrams will shed new light on even familiar calculations. Those with more expertise, who are familiar with notions such as 2-categories, bicategories, and the ideas of formal category theory within them, will hopefully see applications well beyond our stated scope of elementary category theory.

How should you read this book? Chapter 1 introduces many preliminary categorical ideas and examples, laying foundations and fixing notation for later. Readers with some basic familiarity with category theory can probably skip this chapter, returning for details as necessary. String diagrams, the core subject of the book, are introduced in Chapter 2, along with some preliminary examples to start developing intuition for the notation. Chapters 3, 4, and 5 then explore the important topics of monads and adjunctions, using diagrammatic methods.

Along the way, further extensions and perspectives on the graphical notation are introduced and applied. The chapters are probably best read sequentially, as each builds upon the last, allowing us to steadily ramp up to richer examples.

Readers will find we put little emphasis on explicit lemmas and theorems in the text. Instead the focus is on *how* to prove things. As with any mathematics book, it is best read with an accompanying pen and paper, and we suggest that readers draw their own pictures and proofs as they follow along.

The end of each chapter contains a selection of supporting exercises. These have been carefully chosen to reinforce the ideas within the chapter, and readers are encouraged to solve at least the easier questions to check their understanding. To aid the selection of exercises we have made an attempt to indicate the difficulty level or time investment:

2^{2^0}	2^{2^1}	2^{2^2}	2^{2^3}	2^{2^4}
○	⊙	◉	◉	●
2 minutes	5 minutes	quarter of an hour	hours	days

The scheme works on a superexponential scale, ranging from "drill" questions to topics for further research. Of course, the difficulty levels are highly subjective: what some find easy, poses a challenge to others.

In the intended sequel to this monograph, "Exploring String Diagrams—The Art of Category Theory," which we shall tersely refer to as ESD, we plan to delve further into the world of string diagrams. We shall occasionally make forward references to establish connections to ideas intended for ESD, but the current book can be read entirely independently.

How does this connect to applied category theory? Graphical languages are commonplace in applied category theory, with much interest stimulated by the categorical quantum mechanics program (Abramsky and Coecke, 2004).

Opening any textbook in mathematics or the sciences, one quickly encounters diagrams, for example many types of circuits or networks. In conventional texts, they do little more than furnish intuition. A theme of applied category theory is to take these diagrams more seriously, and to exploit them directly in calculations. There are many examples, including quantum theory (Coecke and Kissinger, 2017; Heunen and Vicary, 2019), natural language semantics (Coecke et al., 2010), signal flow graphs (Bonchi et al., 2015), control theory (Baez and Erbele, 2015), economic game theory (Ghani et al., 2018a,b), Markov processes (Baez et al., 2016), analogue (Baez and Fong, 2015) and digital (Ghica and Jung, 2016) electronics and hardware architecture (Brown and Hutton, 1994), machine learning (Fong et al., 2019), and linear algebra (Sobocinski, 2019).

Category theory is the underlying tool upon which all these concrete applications have been built. The current monograph brings graphical methods to bear upon category theory itself. Both users and developers of these diagrammatic languages will find new applications for their graphical intuitions, and hopefully it will also serve as a gateway into the myriad disciplines where graphical reasoning finds a home.

Genesis of the book. The use of string diagrams crept up gradually on the authors. The first author exploited string diagrams in connection to program optimization and monads (Hinze, 2012). At this point both conventional and graphical reasoning are developed in parallel, rather than making the jump to fully diagrammatic arguments.

Meanwhile, the second author was completing a DPhil. in the Quantum group at Oxford, a hotbed for diagrammatic reasoning, and was starting to find string diagrams indispensable for understanding aspects of monad theory. This eventually led to a preliminary account of string diagrammatic elementary category theory (Marsden, 2014, 2015).

Eventually, the two authors noted their common interests, and joined forces, after managing to agree on which way round the diagrams should be drawn! This collaboration led to two papers further fleshing out how effective diagrammatic reasoning can be for elementary category theory (Hinze and Marsden, 2016a,b). This was followed by practical teaching of these ideas by both authors, in the form of invited tutorials at the 21st Estonian Winter School in Computer Science, and QPL 2017.

Throughout this process the authors have learned much about how to best use string diagrams. Many diagrams have been tuned and adjusted to better exploit the notation, with early attempts by both authors being rather naive in their use of the topological freedom afforded by the notation. Eventually our minds, and diagrams, became more flexible as we properly absorbed the capabilities of string diagrams. As their eyes have become open, both authors have also been pleasantly surprised at the sheer breadth of situations in which string diagrammatic arguments prove effective. The current monograph captures our more mature understanding of how string diagrams can best be applied.

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