978-1-009-31507-4 — Nonlife Actuarial Models
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Excerpt
More Information

Part I
Loss Models

In this part of the book we discuss actuarial models for claim losses. The two components of claim losses, namely, claim frequency and claim severity, are modeled separately, and are then combined to derive the aggregateloss distribution. In Chapter 1, we discuss the modeling of claim frequency, introducing some techniques for modeling nonnegative integer-valued random variables. Techniques for modeling continuous random variables relevant for claim severity are discussed in Chapter 2, in which we also consider the effects of coverage modifications on claim frequency and claim severity. Chapter 3 discusses the collective risk model and individual risk model for analyzing aggregate losses. The techniques of convolution and recursive methods are used to compute the aggregate-loss distributions.

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Claim-Frequency Distribution

This book is about modeling the claim losses of insurance policies. Our main interest is nonlife insurance policies covering a fixed period of time, such as vehicle insurance, workers compensation insurance and health insurance. An important measure of claim losses is the claim frequency, which is the number of claims in a block of insurance policies over a period of time. Though claim frequency does not directly show the monetary losses of insurance claims, it is an important variable in modeling the losses.

In this chapter we first briefly review some tools in modeling statistical distributions, in particular, the moment generating function and probability generating function. Some commonly used discrete random variables in modeling claim-frequency distributions, namely, the binomial, geometric, negative binomial and Poisson distributions, are then discussed. We introduce a family of distributions for nonnegative, integer-valued random variables, called the $(a, b, 0)$ class, which includes all the four distributions aforementioned. This class of discrete distributions have found important applications in the actuarial literature. Further methods of creating new nonnegative, integer-valued random variables are introduced. In particular, we discuss the zero-modified distribution, the $(a, b, 1)$ class of distributions, the compound distributions and the mixture distributions.

Learning Objectives

1 Discrete distributions for modeling claim frequency
2 Binomial, geometric, negative binomial and Poisson distributions
3 The $(a, b, 0)$ and $(a, b, 1)$ class of distributions
4 Compound distribution
5 Convolution
6 Mixture distribution

1.1 Claim Frequency, Claim Severity and Aggregate Claim

We consider a block of nonlife insurance policies with coverage over a fixed period of time. The **aggregate claim** for losses of the block of policies is the sum of the monetary losses of all the claims. The number of claims in the block of policies is called the **claim frequency**, and the monetary amount of each claim is called the **claim severity** or **claim size**. A general approach in loss modeling is to consider claim frequency and claim severity separately. The two variables are then combined to model the aggregate claim. Naturally claim frequency is modeled as a nonnegative discrete random variable, while claim severity is continuously distributed.

In this chapter we focus on the claim-frequency distribution. We discuss some nonnegative discrete random variables that are commonly used for modeling claim frequency. Some methods for constructing nonnegative discrete random variables that are suitable for modeling claim frequency are also introduced. As our focus is on short-term nonlife insurance policies, time value of money plays a minor role. We begin with a brief review of some tools for modeling statistical distributions. Further discussions on the topic can be found in the Appendix, as well as the references therein.

1.2 Review of Statistics

Let X be a random variable with **distribution function (df)** $F_X(x)$, which is defined by

$$F_X(x) = \Pr(X \leq x). \tag{1.1}$$

If $F_X(x)$ is a continuous function, X is said to be a **continuous random variable**. Furthermore, if $F_X(x)$ is differentiable, the **probability density function (pdf)** of X, denoted by $f_X(x)$, is defined as

$$f_X(x) = \frac{dF_X(x)}{dx}. \tag{1.2}$$

If X can only take discrete values, it is called a **discrete random variable**. We denote $\Omega_X = \{x_1, x_2, \ldots\}$ as the set of values X can take, called the **support** of X. The **probability function (pf)** of a discrete random variable X, also denoted by $f_X(x)$, is defined as

$$f_X(x) = \begin{cases} \Pr(X = x), & \text{if } x \in \Omega_X, \\ 0, & \text{otherwise.} \end{cases} \tag{1.3}$$

We assume the support of a continuous random variable to be the real line, unless otherwise stated. The rth moment of X about zero (also called the rth raw moment), denoted by $\mathrm{E}(X^r)$, is defined as

$$\mathrm{E}(X^r) = \int_{-\infty}^{\infty} x^r f_X(x)\, dx, \qquad \text{if } X \text{ is continuous,} \tag{1.4}$$

and

$$\mathrm{E}(X^r) = \sum_{x \in \Omega_X} x^r f_X(x), \qquad \text{if } X \text{ is discrete.} \tag{1.5}$$

For convenience, we also write $\mathrm{E}(X^r)$ as μ'_r. The **moment generating function (mgf)** of X, denoted by $M_X(t)$, is a function of t defined by

$$M_X(t) = \mathrm{E}(e^{tX}), \tag{1.6}$$

if the expectation exists. If the mgf of X exists for t in an open interval around $t = 0$, the moments of X exist and can be obtained by successively differentiating the mgf with respect to t and evaluating the result at $t = 0$. We observe that

$$M_X^r(t) = \frac{d^r M_X(t)}{dt^r} = \frac{d^r}{dt^r}\mathrm{E}(e^{tX}) = \mathrm{E}\left[\frac{d^r}{dt^r}(e^{tX})\right] = \mathrm{E}(X^r e^{tX}), \tag{1.7}$$

so that

$$M_X^r(0) = \mathrm{E}(X^r) = \mu'_r. \tag{1.8}$$

If $X_1, X_2, \ldots, X_n$ are **independently and identically distributed (iid)** random variables with mgf $M(t)$, and $X = X_1 + \cdots + X_n$, then the mgf of X is

$$M_X(t) = \mathrm{E}(e^{tX}) = \mathrm{E}(e^{tX_1 + \ldots + tX_n}) = \mathrm{E}\left(\prod_{i=1}^n e^{tX_i}\right) = \prod_{i=1}^n \mathrm{E}(e^{tX_i}) = [M(t)]^n. \tag{1.9}$$

The mgf has the important property that it uniquely defines a distribution. Specifically, if two random variables have the same mgf, their distributions are identical.[1]

If X is a random variable that can only take nonnegative integer values, the **probability generating function (pgf)** of X, denoted by $P_X(t)$, is defined as

$$P_X(t) = \mathrm{E}(t^X), \tag{1.10}$$

if the expectation exists. The mgf and pgf are related through the equations

$$M_X(t) = P_X(e^t), \tag{1.11}$$

and

$$P_X(t) = M_X(\log t). \tag{1.12}$$

Given the pgf of X, we can derive its pf. To see how this is done, note that

$$P_X(t) = \sum_{x=0}^{\infty} t^x f_X(x). \tag{1.13}$$

The rth order derivative of $P_X(t)$ is

$$P_X^r(t) = \frac{d^r}{dt^r}\left(\sum_{x=0}^{\infty} t^x f_X(x)\right) = \sum_{x=r}^{\infty} x(x-1)\cdots(x-r+1)t^{x-r} f_X(x). \tag{1.14}$$

[1] See Appendix A.8 for more details.

If we evaluate $P_X^r(t)$ at $t = 0$, all terms in the above summation vanish except for $x = r$, which is $r!f_X(r)$. Hence, we have

$$P_X^r(0) = r!f_X(r), \tag{1.15}$$

so that given the pgf, we can obtain the pf as

$$f_X(r) = \frac{P_X^r(0)}{r!}. \tag{1.16}$$

In sum, given the mgf of X, the moments of X can be computed through equation (1.8). Likewise, given the pgf of a nonnegative integer-valued random variable, its pf can be computed through equation (1.16). Thus, the mgf and pgf are useful functions for summarizing a statistical distribution.

1.3 Some Discrete Distributions for Claim Frequency

We now review some key results of four discrete random variables, namely, binomial, geometric, negative binomial and Poisson. As these random variables can only take nonnegative integer values, they may be used for modeling the distributions of claim frequency. The choice of a particular distribution in practice is an empirical question to be discussed later.

1.3.1 Binomial Distribution

A random variable X has a binomial distribution with parameters n and θ, denoted by $\mathcal{BN}(n, \theta)$, where n is a positive integer and θ satisfies $0 < \theta < 1$, if the pf of X is

$$f_X(x) = \binom{n}{x}\theta^x(1-\theta)^{n-x}, \qquad \text{for } x = 0, 1, \ldots, n, \tag{1.17}$$

where

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}. \tag{1.18}$$

The mean and variance of X are

$$\mathrm{E}(X) = n\theta \qquad \text{and} \qquad \mathrm{Var}(X) = n\theta(1-\theta), \tag{1.19}$$

so that the variance of X is always smaller than its mean.

The mgf of X is

$$M_X(t) = (\theta e^t + 1 - \theta)^n, \tag{1.20}$$

and its pgf is

$$P_X(t) = (\theta t + 1 - \theta)^n. \tag{1.21}$$

The expression in equation (1.17) is the probability of obtaining x successes in n independent trials each with probability of success θ. The distribution is symmetric if $\theta = 0.5$. It is positively skewed (skewed to the right) if $\theta < 0.5$, and is negatively skewed (skewed to the left) if $\theta > 0.5$. When n is large, X is approximately normally distributed. The convergence to normality is faster the closer θ is to 0.5.

There is a recursive relationship for $f_X(x)$, which can facilitate the computation of the pf. From equation (1.17), we have $f_X(0) = (1-\theta)^n$. Now for $x = 1, \ldots, n$, we have

$$\frac{f_X(x)}{f_X(x-1)} = \frac{\binom{n}{x}\theta^x(1-\theta)^{n-x}}{\binom{n}{x-1}\theta^{x-1}(1-\theta)^{n-x+1}} = \frac{(n-x+1)\theta}{x(1-\theta)}, \tag{1.22}$$

so that

$$f_X(x) = \left[\frac{(n-x+1)\theta}{x(1-\theta)}\right] f_X(x-1). \tag{1.23}$$

Example 1.1 Plot the pf of the binomial distribution for $n = 10$, and $\theta = 0.2$, 0.4, 0.6 and 0.8.

Solution 1.1 Figure 1.1 plots the pf of $\mathcal{BN}(n, \theta)$ for $\theta = 0.2$, 0.4, 0.6 and 0.8, with $n = 10$.

It can be clearly seen that the binomial distribution is skewed to the right for $\theta = 0.2$ and skewed to the left for $\theta = 0.8$. □

1.3.2 Geometric Distribution

A nonnegative discrete random variable X has a geometric distribution with parameter θ for $0 < \theta < 1$, denoted by $\mathcal{GM}(\theta)$, if its pf is given by

$$f_X(x) = \theta(1-\theta)^x, \qquad \text{for } x = 0, 1, \ldots. \tag{1.24}$$

The mean and variance of X are

$$\mathrm{E}(X) = \frac{1-\theta}{\theta} \qquad \text{and} \qquad \mathrm{Var}(X) = \frac{1-\theta}{\theta^2}, \tag{1.25}$$

so that, in contrast to the binomial distribution, the variance of a geometric distribution is always larger than its mean.

The expression in equation (1.24) is the probability of having x failures prior to the first success in a sequence of independent Bernoulli trials with probability of success θ.

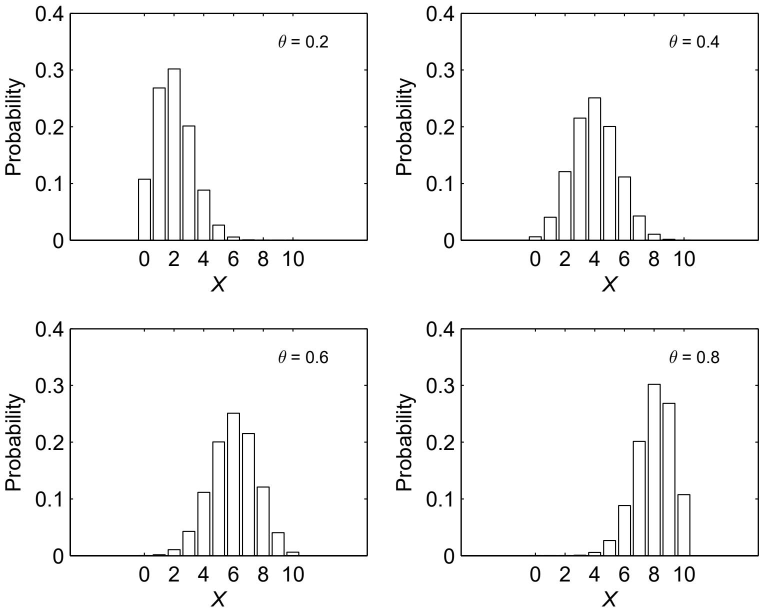

Figure 1.1 Probability function of $\mathcal{BN}(10, \theta)$

The mgf of X is

$$M_X(t) = \frac{\theta}{1-(1-\theta)e^t}, \tag{1.26}$$

and its pgf is

$$P_X(t) = \frac{\theta}{1-(1-\theta)t}. \tag{1.27}$$

The pf of X is decreasing in x. It satisfies the following recursive relationship

$$f_X(x) = (1-\theta)\, f_X(x-1), \tag{1.28}$$

for $x = 1, 2, \ldots$, with starting value $f_X(0) = \theta$.

1.3.3 Negative Binomial Distribution

A nonnegative discrete random variable X has a negative binomial distribution with parameters r and θ, denoted by $\mathcal{NB}(r, \theta)$, if the pf of X is

$$f_X(x) = \binom{x+r-1}{r-1} \theta^r (1-\theta)^x, \qquad \text{for } x = 0, 1, \ldots, \tag{1.29}$$

where r is a positive integer and θ satisfies $0 < \theta < 1$. The geometric distribution is a special case of the negative binomial distribution with $r = 1$. We may interpret the expression in equation (1.29) as the probability of getting x failures prior to the rth success in a sequence of independent Bernoulli trials with probability of success θ. Thus, $\mathcal{NB}(r, \theta)$ is just the sum of r independently distributed $\mathcal{GM}(\theta)$ variates. Hence, using equation (1.25), we can conclude that if X is distributed as $\mathcal{NB}(r, \theta)$ its mean and variance are

$$\mathrm{E}(X) = \frac{r(1-\theta)}{\theta} \quad \text{and} \quad \mathrm{Var}(X) = \frac{r(1-\theta)}{\theta^2}, \tag{1.30}$$

so that its variance is always larger than its mean.

Furthermore, using the results in equations (1.9), (1.26) and (1.27), we obtain the mgf of $\mathcal{NB}(r, \theta)$ as

$$M_X(t) = \left[\frac{\theta}{1-(1-\theta)e^t}\right]^r, \tag{1.31}$$

and its pgf as

$$P_X(t) = \left[\frac{\theta}{1-(1-\theta)t}\right]^r. \tag{1.32}$$

Note that the binomial coefficient in equation (1.29) can be written as

$$\begin{aligned}\binom{x+r-1}{r-1} &= \frac{(x+r-1)!}{(r-1)!x!} \\ &= \frac{(x+r-1)(x+r-2)\cdots(r+1)r}{x!}.\end{aligned} \tag{1.33}$$

The expression in the last line of the above equation is well defined for any number $r > 0$ (not necessarily an integer) and any nonnegative integer x.[2] Thus, if we define

$$\binom{x+r-1}{r-1} = \frac{(x+r-1)(x+r-2)\cdots(r+1)r}{x!}, \tag{1.34}$$

we can use equation (1.29) as a pf even when r is not an integer. Indeed it can be verified that

$$\sum_{x=0}^{\infty} \binom{x+r-1}{r-1} \theta^r (1-\theta)^x = 1, \tag{1.35}$$

for $r > 0$ and $0 < \theta < 1$, so that the extension of the parameter r of the negative binomial distribution to any positive number is meaningful. We shall adopt this extension in any future applications.

The recursive formula of the pf follows from the result

$$\frac{f_X(x)}{f_X(x-1)} = \frac{\binom{x+r-1}{r-1}\theta^r(1-\theta)^x}{\binom{x+r-2}{r-1}\theta^r(1-\theta)^{x-1}} = \frac{(x+r-1)(1-\theta)}{x}, \tag{1.36}$$

[2] As factorials are defined only for nonnegative integers, the expression in the first line of equation (1.33) is not defined if r is not an integer.

Table 1.1. *Results of Example 1.2*

x	$r = 0.5$	$r = 1.0$	$r = 1.5$	$r = 2.0$
0	0.6325	0.4000	0.2530	0.1600
1	0.1897	0.2400	0.2277	0.1920
2	0.0854	0.1440	0.1708	0.1728
3	0.0427	0.0864	0.1195	0.1382

so that

$$f_X(x) = \left[\frac{(x + r - 1)(1 - \theta)}{x}\right] f_X(x - 1), \tag{1.37}$$

with starting value

$$f_X(0) = \theta^r. \tag{1.38}$$

Example 1.2 Using the recursion formula, calculate the pf of the negative binomial distribution with $r = 0.5$, 1, 1.5 and 2, and $\theta = 0.4$, for $x = 0$, 1, 2 and 3. What is the mode of the negative binomial distribution?

Solution 1.2 From the recursion formula in equation (1.37), we have

$$f_X(x) = \left[\frac{0.6(x + r - 1)}{x}\right] f_X(x - 1), \qquad \text{for } x = 1, 2, \ldots,$$

with starting value $f_X(0) = (0.4)^r$. We summarize the results in Table 1.1.

Note that the modes for $r = 0.5$, 1 and 1.5 are 0, and that for $r = 2$ is 1. To compute the mode in general, we note that, from equation (1.37),

$$f_X(x) > f_X(x - 1) \qquad \text{if and only if} \qquad \frac{(x + r - 1)(1 - \theta)}{x} > 1,$$

and the latter inequality is equivalent to

$$x < \frac{(r - 1)(1 - \theta)}{\theta}.$$

Therefore, the mode of the negative binomial distribution is equal to the non-negative integer part of $(r - 1)(1 - \theta)/\theta$. We can verify this result from Table 1.1. For example, when $r = 2$,

$$\frac{(r - 1)(1 - \theta)}{\theta} = \frac{0.6}{0.4} = 1.5,$$

and its integer part (the mode) is 1. □