Graph Theory and Additive Combinatorics

Using the dichotomy of structure and pseudorandomness as a central theme, this accessible text provides a modern introduction to extremal graph theory and additive combinatorics. Readers will explore central results in additive combinatorics – notably the cornerstone theorems of Roth, Szemerédi, Freiman, and Green–Tao – and will gain additional insights into these ideas through graph theoretic perspectives. Topics discussed include the Turán problem, Szemerédi's graph regularity method, pseudorandom graphs, graph limits, graph homomorphism inequalities, Fourier analysis in additive combinatorics, the structure of set addition, and the sum-product problem. Important combinatorial, graph theoretic, analytic, Fourier, algebraic, and geometric methods are highlighted. Students will appreciate the chapter summaries, many figures and exercises, and freely available lecture videos on MIT OpenCourseWare. Meant as an introduction for students and researchers studying combinatorics, theoretical computer science, analysis, probability, and number theory, the text assumes only basic familiarity with abstract algebra, analysis, and linear algebra.

Yufei Zhao is Associate Professor of Mathematics at the Massachusetts Institute of Technology. His research tackles a broad range of problems in discrete mathematics, including extremal, probabilistic, and additive combinatorics, graph theory, and discrete geometry, as well as applications to computer science. His honors include the SIAM Dénes Kõnig prize (2018), the Sloan Research Fellowship (2019), and the NSF CAREER Award (2021). This book is based on an MIT graduate course, which he has taught and developed over the last five years.

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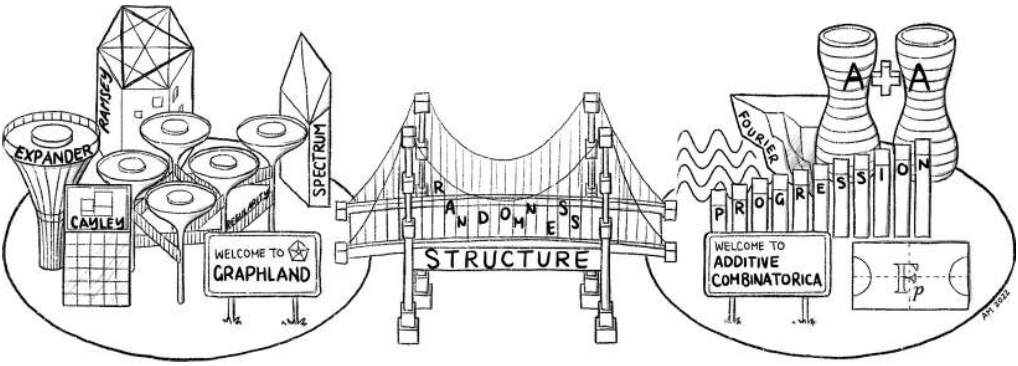
Exploring Structure and Randomness

Yufei Zhao
Massachusetts Institute of Technology



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To Lu
for your constant love and support
and Andi
who arrived in time to get on this page

Contents

Preface

Who Is This Book For?

This textbook is intended for graduate and advanced undergraduate students as well as researchers in mathematics, computer science, and related areas. The material should appeal to anyone with an interest in combinatorics, theoretical computer science, analysis, probability, and number theory. It can be used as a textbook for a class or self-study, or as a research reference.

Why This Book?

There have been many exciting developments in graph theory and additive combinatorics in recent decades. This is the first introductory graduate-level textbook to focus on a unifying set of topics connecting graph theory and additive combinatorics.

This textbook arose from a one-semester graduate-level course that I developed at Massachusetts Institute of Technology (and still teach regularly) to introduce students to a spectrum of beautiful mathematics in the field.

Lecture Videos

A complete set of video lectures from my Fall 2019 class is available for free through MIT OpenCourseWare and YouTube (search for *Graph Theory and Additive Combinatorics* and *MIT OCW*). The lecture videos are a useful resource and complement this book.

What Is This Book About?

This book introduces the readers to classical and modern developments in graph theory and additive combinatorics, with a focus on topics and themes that connect the two subjects.

A foundational result in additive combinatorics is **Roth's theorem**, which says that every subset of $\{1, 2, \ldots, \}$ without a 3-term arithmetic progression has at most $o(N)$ elements. We will see different proofs of Roth's theorem, using tools from graph theory and Fourier analysis. A key idea in both approaches is the *dichotomy of structure versus pseudorandomness*.

Roth's theorem laid the groundwork for many important later developments, such as

- **Szemerédi's theorem:** Every set of integers of positive density contains arbitrarily long arithmetic progressions; and
- **Green–Tao theorem:** The primes contain arbitrarily long arithmetic progressions.

A core thread throughout the book is the connection bridging graph theory and additive combinatorics. The book opens with Schur's theorem, which is an early example whose proof illustrates this connection. Graph theoretic perspectives are presented throughout the book.

Here are some of the topics and questions considered in this book:

Chapter 1: **Forbidding a subgraph**
What is the maximum number of edges in a triangle-free graph on n vertices? What if instead we forbid some other subgraph? This is known as the Turán problem.

Chapter 2: **Graph regularity method**
Szemerédi introduced this powerful tool that provides an approximate structural description for every large graph.

Chapter 3: **Pseudorandom graphs**
What does it mean for some graph to resemble a random graph?

Chapter 4: **Graph limits**
In what sense can a sequence of graphs, increasing in size, converge to some limit object?

Chapter 5: **Graph homomorphism inequalities**
What are possible relationships between subgraph densities?

Chapter 6: **Forbidding a 3-term arithemtic progression**
Roth's theorem and Fourier analysis in additive combinatorics.

Chapter 7: **Structure of set addition**
What can one say about a set of integer A with small sumset $A + A = \{a + b : a, b \in A\}$? Freiman's theorem is a foundational result that gives an answer.

Chapter 8: **Sum-product problem**
Can a set A simultaneously have both small sumset $A + A$ and product set $A \cdot A$?

Chapter 9: **Progressions in sparse pseudorandom sets**
Key ideas in the proof of the Green–Tao theorem. How can we apply a dense setting result, namely Szemerédi's theorem, to a sparse set?

For a more detailed list of topics, see the highlights and summary boxes at the beginning and the end of each chapter.

The book is roughly divided into two parts, with graph theory the focus of Chapters 1 to 5 and additive combinatorics the focus of Chapters 6 to 9. These are not disjoint and separate subjects. Rather, graph theory and additive combinatorics are interleaved throughout the book. We emphasize their interactions. Each chapter can be enjoyed independently, as there are very few dependencies between chapters, though one gets the most out of the book by appreciating the connections.

Using the Textbook for a Class

The contents may be taught as a fast-paced one-semester class, or as a two-semester sequence, with each term focusing on one half of the book: the first on extremal graph theory, and the second on additive combinatorics.

For a one-semester class (which is how I teach it at MIT; see my website or MIT OCW for syllabus, lecture videos, homework, and further information), I suggest skipping some more technical or advanced topics and proofs, such as the following: (Chapter 1) the proofs of the Erdős–Stone–Simonovits theorem, the $K_{s,t}$ construction, randomized algebraic construction; (Chapter 2) the proof of the graph counting lemma, induced graph removal and strong regularity, hypergraph regularity and removal; (Chapter 3) quasirandom groups, quasirandom Cayley graphs; (Chapter 4) most technical proofs on graph limits; (Chapter 5) Hölder, entropy; (Chapter 6) arithmetic regularity and popular common difference; (Chapter 7) proofs later in the chapter if short on time; (Chapter 9) proof details.

For a class focused on one part of the book, one may wish to explore further topics as suggested in *Further Reading* at the end of each chapter.

Prerequisites

The prerequisites are minimal – primarily mathematical maturity and an interest in combinatorics. Some basic concepts from abstract algebra, analysis, and linear algebra are assumed.

Exercises

The book contains around 150 carefully selected exercises. They are scattered throughout each chapter. Some exercises are embedded in the middle of a section – these exercises are meant as routine tests of understanding of the concepts just discussed. For example, they sometimes ask you to fill in missing proof details or think about easy generalizations and extensions. The exercises at the end of each section are carefully selected problems that reinforce the techniques discussed in the chapter. Hopefully they are all interesting. Most of them are intended to test your mastery of the techniques taught in the chapter. Many of these end-of-chapter exercises are quite challenging, with starred problems intended to be more difficult but still doable by a strong student given the techniques taught. Many of these exercises are adapted from lemmas and results from research papers. (I apologize for omitting references for the exercises so that they can be used as homework assignments.)

Spending time with the exercises is essential for mastering the techniques. I used many of these exercises in my classes. My students often told me that they thought that they had understood the material after a lecture, only to discover their incomplete mastery when confronted with the exercises. Struggling with these exercises led them to newfound insight.

Further Reading

This is a massive and rapidly expanding subject. The book is intended to be introductory and enticing rather than comprehensive. Each chapter concludes with recommendations for further reading for anyone who wishes to learn more. Additionally, references are given generously throughout the text for anyone who wishes to dive deeper and read the original sources.

Acknowledgments

I thank all my teachers and mentors who have taught me the subject starting from when I was a graduate student, with a special shout-out to my Ph.D. advisor Jacob Fox for his dedicated mentorship. I first encountered this subject at the University of Cambridge, when I took a Part III class on extremal graph theory taught by David Conlon. Over the years, I learned a lot from various researchers thanks to their carefully and insightfully written lecture notes scattered on the web, in particular those by David Conlon, Tim Gowers, Andrew Granville, Ben Green, Choongbum Lee, László Lovász, Imre Ruzsa, Asaf Shapira, Adam Sheffer, K. Soundararajan, Terry Tao, and Jacques Verstraete.

This book arose from a one-semester course that I taught at MIT in Fall 2017, 2019, and 2021. I thank all my amazing and dedicated students who kept up their interest in my teaching – they were instrumental in motivating me to complete this book project. Students from the 2017 and 2019 classes took notes based on my lectures, which I subsequently rewrote and revised into this book. My 2021 class used an early draft of this book and gave valuable comments and feedback. There are many students whom I wish to thank, and here is my attempt at listing them (my apologies to anyone whose name I have inadvertently omitted): Dhroova Aiylam, Ganesh Ajjanagadde, Shyan Akmal, Ryan Alweiss, Morris Ang Jie Jun, Adam Ardeishar, Matt Babbitt, Yonah Borns-Weil, Matthew Brennan, Brynmor Chapman, Evan Chen, Byron Chin, Ahmed Chowdhury Zawad, Anlong Chua, Travis Dillon, Jonathan Figueroa Rodríguez, Christian Gaetz, Shengwen Gan, Jiyang Gao, Yibo Gao, Swapnil Garg, Benjamin Gunby, Meghal Gupta, Kaarel Haenni, Milan Haiman, Linus Hamilton, Carina Hong Letong, Vishesh Jain, Pakawut Jiradilok, Sujay Kazi, Dain Kim, Elena Kim, Younhun Kim, Yael Kirkpatrick, Daishi Kiyohara, Frederic Koehler, Keiran Lewellen, Anqi Li, Jerry Li, Allen Liu, Michael Ma, Nitya Mani, Olga Medrano, Holden Mui, Eshaan Nichani, Yuchong Pan, Minjae Park, Alan Peng, Saranesh Prembabu, Michael Ren, Dhruv Rohatgi, Diego Roque, Ashwin Sah, Maya Sankar, Mehtaab Sawhney, Carl Schildkraut, Tristan Shin, Mihir Singhal, Tomasz Slusarczyk, Albert Soh, Kevin Sun, Sarah Tammen, Jonathan Tidor, Paxton Turner, Danielle Wang, Hong Wang, Nicole Wein, Jake Wellens, Chris Xu, Max Wenqiang Xu, Yinzhan Xu, Zixuan Xu, Lisa Yang, Yuan Yao, Richard Yi, Hung-Hsun Yu, Lingxian Zhang, Kai Zheng, Yunkun Zhou. Additionally, I would like to thank Thomas Bloom and Zilin Jiang for carefully reading the book draft and sending in many suggestions for corrections and improvements.

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Notation and Conventions

We use standard notation in this book. The comments here are mostly for clarification. You might want to skip this section and return to it only as needed.

Sets

We write $\boldsymbol{[N]} := \{1, 2, \dots, N\}$. Also $\mathbb{N} := \{1, 2, \dots\}$.

Given a finite set S and a positive integer r, we write $\binom{S}{r}$ for the set of r-element subsets of S.

If S is a finite set and f is a function on S, we use the expectation notation $\mathbb{E}_{x \in S} f(x)$, or more simply $\mathbb{E}_x f(x)$ (or even $\mathbb{E} f$ if there is no confusion) to mean the average $|S|^{-1} \sum_{x \in S} f(x)$. We also use the symbol $\mathbb{E}$ for its usual meaning as the expectation for some random variable.

A ***k-term arithmetic progression*** (abbreviated ***k-AP***) in an abelian group is a sequence of the form

$$a, a + d, a + 2d, \dots, a + (k - 1)d.$$

Here d is called the common difference. The progression is called ***nontrivial*** if $d \neq 0$, and ***trivial*** if $d = 0$. When we say that a set A contains a k-AP, we mean that it contains a nontrivial k-AP. Likewise, when we say that A is ***k-AP-free***, we mean that it contains no nontrivial k-APs.

Graphs

We write a graph as $\boldsymbol{G = (V, E)}$, where V is a finite set of vertices, and E is the set of edges. Each edge is an unordered pair of distinct vertices. Formally, $E \subseteq \binom{V}{2}$.

Given a graph G, we write $\boldsymbol{V(G)}$ for the set of vertices, and $\boldsymbol{E(G)}$ for the set of edges, and denote their cardinalities by $\boldsymbol{v(G)} := |V(G)|$ and $\boldsymbol{e(G)} := |E(G)|$.

In a graph G, the ***neighborhood*** of a vertex x, denoted $\boldsymbol{N_G(x)}$ (or simply $N(x)$ if there is no confusion), is the set of vertices y such that xy is an edge. The ***degree*** of x is the number of neighbors of x, denoted $\mathbf{deg}_G(x) := |N_G(x)|$ (or simply written as $\mathbf{deg}(x)$).

Given a graph G, for each $A \subseteq V(G)$, we write $\boldsymbol{e(A)}$ to denote the number of edges with both endpoints in A. Given $A, B \subseteq V(G)$ (not necessarily disjoint), we write

$$\boldsymbol{e(A, B)} := |\{(a, b) \in A \times B : ab \in E(G)\}| .$$

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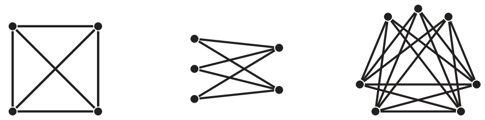
Note that when A and B are disjoint, $e(A, B)$ is the number of the edges between A and B. On the other hand, $e(A, A) = 2e(A)$ as each edge within A is counted twice.

Here are some standard graphs:

- $\boldsymbol{K_r}$ is the complete graph on r vertices, also known as an ***r-clique***;
- $\boldsymbol{K_{s,t}}$ is the complete bipartite graph with s vertices in one vertex part and t vertices in the other vertex part;
- $\boldsymbol{K_{r,s,t}}$ is a complete tripartite graph with vertex parts having sizes r, s, t respectively (e.g., $K_{1,1,1} = K_3$); and so on analogously for complete multipartite graphs with more parts;
- $\boldsymbol{C_\ell}$ ($\ell \geq 3$) is a cycle with ℓ vertices and ℓ edges.

Some examples are shown below.

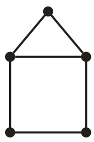


K_4 $K_{3,2}$ $K_{3,2,2}$ C_8

Given two graphs H and G, we say that H is a ***subgraph*** of G if one can delete some vertices and edges from G to obtain a graph isomorphic to H. A ***copy*** of H in G is a subgraph G that is isomorphic to H. A ***labeled copy*** of H in G is a subgraph of G isomorphic to H where we also specify the isomorphism from H. Equivalently, a labeled copy of H in G is an injective graph homomorphism from H to G. For example, if G has q copies of K_3, then G has $6q$ labeled copies of K_3.

We say that H is an ***induced subgraph*** of G if one can delete some vertices of G (when we delete a vertex, we also remove all edges incident to the vertex) to obtain H – note that in particular we are not allowed to remove additional edges other than those incident to a deleted vertex. If $S \subseteq V(G)$, we write $\boldsymbol{G[S]}$ to denote the subgraph of G induced by the vertex set S, that is, $G[S]$ is the subgraph with vertex set S and keeping all the edges from G among S.

As an example, the following graph contains the 4-cycle as an induced subgraph. It contains the 5-cycle as a subgraph but not as an induced subgraph.



In this book, when we say ***H-free***, we always mean not containing H as a subgraph. On the other hand, we say ***induced H-free*** to mean not containing H as an induced subgraph.

Given two graphs F and G, a ***graph homomorphism*** is a map $\phi\colon V(F) \to V(G)$ (not necessarily injective) such that $\phi(u)\phi(v) \in E(G)$ whenever $uv \in E(F)$. In other words, ϕ is a map of vertices that sends edges to edges. A key difference between a copy of F in G and a graph homomorphism from F to G is that the latter does not have to be an injective map of vertices.

The ***chromatic number*** $\chi(G)$ of a graph G is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices receive the same color (such a coloring is called a ***proper coloring***).

The ***adjacency matrix*** of a graph $G = (V, E)$ is a $v(G) \times v(G)$ matrix whose rows and columns both are indexed by V, and such that the entry indexed by $(u, v) \in V \times V$ is 1 if $uv \in E$ and 0 if $uv \notin E$.

An ***r-uniform hypergraph*** (also called ***r-graph*** for short) consists of a finite vertex set V along with an edge set $E \subseteq \binom{V}{r}$. Each edge of the r-graph is an r-element subset of vertices.

Asymptotics

We use the following standard asymptotic notation. Given nonnegative quantities f and g, in each of the following items, the various notations have the same meaning (as some parameter, usually n, tends to infinity):

- $\boldsymbol{f \lesssim g}$, $\boldsymbol{f = O(g)}$, $\boldsymbol{g = \Omega(f)}$, $f \le Cg$ for some constant $C > 0$
- $\boldsymbol{f = o(g)}$, $f/g \to 0$
- $\boldsymbol{f = \Theta(g)}$, $\boldsymbol{f \asymp g}$, $g \lesssim f \lesssim g$
- $\boldsymbol{f \sim g}$, $f = (1 + o(1))g$

Subscripts (e.g., $\boldsymbol{O_s(\ )}$, $\boldsymbol{\lesssim_s}$), are used to emphasize that the hidden constants may depend on the subscripted parameters. For example, $f(s, x) \lesssim_s g(s, x)$ means that for every s there is some constant C_s so that $f(s, x) \le C_s g(s, x)$ for all x.

We avoid using $\ll$ since this notation carries different meanings in different communities and by different authors. In analytic number theory, $f \ll g$ is standard for $f = O(g)$ (this is called Vinogradov notation). In combinatorics and probability, $f \ll g$ sometimes means $f = o(g)$, and sometimes means that f is sufficiently small depending on g.

When asymptotic notation is used in the hypothesis of a statement, it should be interpreted as being applied to a sequence rather than a single object. For example, given functions f and g, we write

$$\textit{if } f(G) = o(1), \textit{ then } g(G) = o(1)$$

to mean

$$\textit{if a sequence } G_n \textit{ satisfies } f(G_n) = o(1), \textit{ then } g(G_n) = o(1),$$

which is also equivalent to

$$\textit{for every } \varepsilon > 0 \textit{ there is some } \delta > 0 \textit{ such that, if } |f(G)| \le \delta, \textit{ then } |g(G)| \le \varepsilon.$$