CHAPTER

1 Stern-Gerlach Experiments

It was not a dark and stormy night when Otto Stern and Walther Gerlach performed their now famous experiment in 1922. The Stern-Gerlach experiment demonstrated that measurements on microscopic or quantum particles are not always as certain as we might expect. Quantum particles behave as mysteriously as Erwin's socks—sometimes forgetting what we have already measured. Erwin's adventure with the mystery socks is farfetched because you know that everyday objects do not behave like his socks. If you observe a sock to be black, it remains black no matter what other properties of the sock you observe. However, the Stern-Gerlach experiment goes against these ideas. Microscopic or quantum particles do not behave like the classical objects of your everyday experience. The act of observing a quantum particle affects its measurable properties in a way that is foreign to our classical experience.

In these first three chapters, we focus on the Stern-Gerlach experiment because it is a conceptually simple experiment that demonstrates many basic principles of quantum mechanics. We discuss a variety of experimental results and the quantum theory that has been developed to predict those results. The mathematical formalism of quantum mechanics is based upon six postulates that we will introduce as we develop the theoretical framework. (A complete list of these postulates is in Section 1.5.) We use the Stern-Gerlach experiment to learn about quantum mechanics theory for two primary reasons: (1) It demonstrates how quantum mechanics works in principle by illustrating the postulates of quantum mechanics, and (2) it demonstrates how quantum mechanics works in practice through the use of Dirac notation and matrix mechanics to solve problems. By using a simple example, we can focus on the principles and the new mathematics, rather than having the complexity of the physics obscure these new aspects.

1.1 ■ STERN-GERLACH EXPERIMENT

In 1922 Otto Stern and Walther Gerlach performed a seminal experiment in the history of quantum mechanics. In its simplest form, the experiment consisted of an oven that produced a beam of neutral atoms, a region of space with an inhomogeneous magnetic field, and a detector for the atoms, as depicted in Fig. 1.1. Stern and Gerlach used a beam of silver atoms and found that the beam was split into two in its passage through the magnetic field. One beam was deflected upwards and one downwards in relation to the direction of the magnetic field gradient.

To understand why this result is so at odds with our classical expectations, we must first analyze the experiment classically. The results of the experiment suggest an interaction between a neutral particle and a magnetic field. We expect such an interaction if the particle possesses a magnetic moment $\boldsymbol{\mu}$. The potential energy of this interaction is $E = -\boldsymbol{\mu}\cdot\mathbf{B}$, which results in a force $\mathbf{F} = \nabla(\boldsymbol{\mu}\cdot\mathbf{B})$. In the

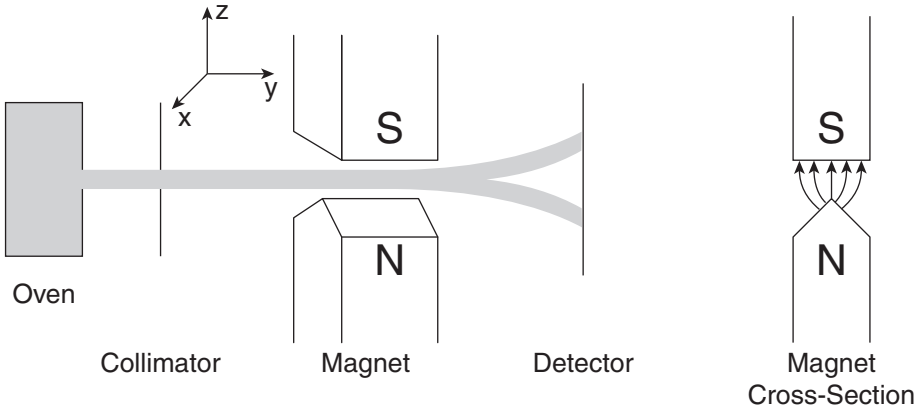

FIGURE 1.1 Stern-Gerlach experiment to measure the spin component of neutral particles along the z-axis. The magnet cross section at right shows the inhomogeneous field used in the experiment.

Stern-Gerlach experiment, the magnetic field gradient is primarily in the z-direction, and the resulting z-component of the force is

$$\begin{aligned} F_z &= \frac{\partial}{\partial z}(\boldsymbol{\mu}\cdot\mathbf{B}) \\ &\cong \mu_z \frac{\partial B_z}{\partial z}. \end{aligned} \tag{1.1}$$

This force is perpendicular to the direction of motion and deflects the beam in proportion to the component of the magnetic moment in the direction of the magnetic field gradient.

Now consider how to understand the origin of the atom's magnetic moment from a classical viewpoint. The atom consists of charged particles, which, if in motion, can produce loops of current that give rise to magnetic moments. A loop of area A and current I produces a magnetic moment

$$\mu = IA \tag{1.2}$$

in MKS units. If this loop of current arises from a charge q traveling at speed v in a circle of radius r, then

$$\begin{aligned} \mu &= \frac{q}{2\pi r/v}\pi r^2 \\ &= \frac{qrv}{2} \\ &= \frac{q}{2m}L, \end{aligned} \tag{1.3}$$

where $L = mrv$ is the orbital angular momentum of the particle. In the same way that the earth revolves around the sun and rotates around its own axis, we can also imagine a charged particle in an atom having **orbital angular momentum L** and a new property, the **intrinsic angular momentum**, which we label **S** and call **spin**. The intrinsic angular momentum also creates current loops, so we expect a similar relation between the magnetic moment $\boldsymbol{\mu}$ and **S**. The exact calculation

involves an integral over the charge distribution, which we will not do. We simply assume that we can relate the magnetic moment to the intrinsic angular momentum in the same fashion as Eq. (1.3), giving

$$\boldsymbol{\mu} = g\frac{q}{2m}\mathbf{S}, \tag{1.4}$$

where the dimensionless **gyromagnetic ratio** g contains the details of that integral.

A silver atom has 47 electrons, 47 protons, and 60 or 62 neutrons (for the most common isotopes). The magnetic moments depend on the inverse of the particle mass, so we expect the heavy protons and neutrons ($\approx 2000\, m_e$) to have little effect on the magnetic moment of the atom and so we neglect them. From your study of the periodic table in chemistry, you recall that silver has an electronic configuration $1s^2 2s^2 2p^6 3s^2 3p^6 4s^2 3d^{10} 4p^6 4d^{10} 5s^1$, which means that there is only the lone $5s$ electron outside of the closed shells. The electrons in the closed shells can be represented by a spherically symmetric cloud with no orbital or intrinsic angular momentum (unfortunately we are injecting some quantum mechanical knowledge of atomic physics into this classical discussion). That leaves the lone $5s$ electron as a contributor to the magnetic moment of the atom as a whole. An electron in an s state has no orbital angular momentum, but it does have spin. Hence the magnetic moment of this electron, and therefore of the entire neutral silver atom, is

$$\boldsymbol{\mu} = -g\frac{e}{2m_e}\mathbf{S}, \tag{1.5}$$

where e is the magnitude of the electron charge. The classical force on the atom can now be written as

$$F_z \cong -g\frac{e}{2m_e}S_z\frac{\partial B_z}{\partial z}. \tag{1.6}$$

The deflection of the beam in the Stern-Gerlach experiment is thus a measure of the component (or projection) S_z of the spin along the z-axis, which is the orientation of the magnetic field gradient.

If we assume that the $5s$ electron of each atom has the same magnitude $|\mathbf{S}|$ of the intrinsic angular momentum or spin, then classically we would write the z-component as $S_z = |\mathbf{S}|\cos\theta$, where θ is the angle between the z-axis and the direction of the spin $\mathbf{S}$. In the thermal environment of the oven, we expect a random distribution of spin directions and hence all possible angles θ. Thus we expect some continuous distribution (the details are not important) of spin components from $S_z = -|\mathbf{S}|$ to $S_z = +|\mathbf{S}|$, which would yield a continuous spread in deflections of the silver atomic beam. Rather, the experimental result that Stern and Gerlach observed was that there are only two deflections, indicating that there are only two possible values of the z-component of the electron spin. The magnitudes of these deflections are consistent with values of the spin component of

$$S_z = \pm\frac{\hbar}{2}, \tag{1.7}$$

where $\hbar$ is Planck's constant h divided by 2π and has the numerical value

$$\begin{aligned}\hbar &= 1.0546 \times 10^{-34}\ \text{J•s} \\ &= 6.5821 \times 10^{-16}\ \text{eV•s}.\end{aligned} \tag{1.8}$$

This result of the Stern-Gerlach experiment is evidence of the **quantization** of the electron's spin angular momentum component along an axis. This quantization is at odds with our classical

expectations for this measurement. The factor of 1/2 in Eq. (1.7) leads us to refer to this as a **spin-1/2** system.

In this example, we have chosen the z-axis along which to measure the spin component, but there is nothing special about this direction in space. We could have chosen any other axis and we would have obtained the same results.

Now that we know the fine details of the Stern-Gerlach experiment, we simplify the experiment for the rest of our discussions by focusing on the essential features. A simplified schematic representation of the experiment is shown in Fig. 1.2, which depicts an oven that produces the beam of atoms, a Stern-Gerlach device with two output ports for the two possible values of the spin component, and two counters to detect the atoms leaving the output ports of the Stern-Gerlach device. The Stern-Gerlach device is labeled with the axis along which the magnetic field is oriented. The up and down arrows indicate the two possible measurement results for the device; they correspond respectively to the results $S_z = \pm\hbar/2$ in the case where the field is oriented along the z-axis. There are only two possible results in this case, so they are generally referred to as **spin up** and **spin down**. The physical quantity that is measured, S_z in this case, is called an **observable**. In our detailed discussion of the experiment above, we chose the field gradient in such a manner that the spin up states were deflected upwards. In this new simplification, the deflection itself is not an important issue. We simply label the output port with the desired state and count the particles leaving that port. The Stern-Gerlach device sorts (or filters, selects or analyzes) the incoming particles into the two possible outputs $S_z = \pm\hbar/2$ in the same way that Erwin sorted his socks according to color or length. We follow convention and refer to a Stern-Gerlach device as an **analyzer**.

In Fig. 1.2, the input and output beams are labeled with a new symbol called a **ket**. We use the ket $|+\rangle$ as a mathematical representation of the quantum state of the atoms that exit the upper port corresponding to $S_z = +\hbar/2$. The lower output beam is labeled with the ket $|-\rangle$, which corresponds to $S_z = -\hbar/2$, and the input beam is labeled with the more generic ket $|\psi\rangle$. The kets are representations of the quantum states. They are used in mathematical expressions and they represent all the information that we can know about the state. This ket notation was developed by Paul A. M. Dirac and is central to the approach to quantum mechanics that we take in this text. We will discuss the mathematics of these kets in full detail later. With regard to notation, you will find many different ways of writing the same ket. The symbol within the ket brackets is any simple label to distinguish the ket from other different kets. For example, the kets $|+\rangle$, $|+\hbar/2\rangle$, $|S_z = +\hbar/2\rangle$, $|+\hat{\mathbf{z}}\rangle$, and $|\uparrow\rangle$ are all equivalent ways of writing the same thing, which in this case signifies that we have measured the z-component of the spin and found it to be $+\hbar/2$ or spin up. Though we may label these kets in different ways, they all refer to the same physical state and so they all behave the same mathematically. The symbol $|\pm\rangle$ refers to both the $|+\rangle$ and $|-\rangle$ kets. The first postulate of quantum mechanics tells us that kets in general describe the quantum state mathematically and that they contain all the information that we can know about the state. We denote a general ket as $|\psi\rangle$.

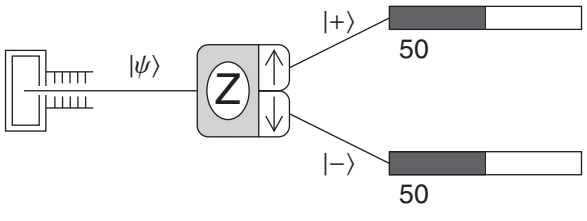

FIGURE 1.2 Simplified schematic of the Stern-Gerlach experiment, depicting a source of atoms, a Stern-Gerlach analyzer, and two counters.

Postulate 1

The state of a quantum mechanical system, including all the information you can know about it, is represented mathematically by a normalized ket $|\psi\rangle$.

We have chosen the particular simplified schematic representation of the Stern-Gerlach experiment shown in Fig. 1.2, because it is the same representation used in the SPINS software program that you may use to simulate these experiments. The SPINS program allows you to perform all the experiments described in this text. This software is freely available, as detailed in Resources at the end of the chapter. In the SPINS program, the components are connected with simple lines to represent the paths the atoms take. The directions and magnitudes of deflections of the beams in the program are not relevant. That is, whether the spin up output beam is drawn as deflected upwards, downwards, or not at all, is not relevant. The labeling on the output port is enough to tell us what that state is. Thus the extra ket label $|+\rangle$ on the spin up output beam in Fig. 1.2 is redundant and will be dropped soon.

The SPINS program permits alignment of Stern-Gerlach analyzing devices along all three axes and also at any angle ϕ measured from the x-axis in the x-y plane. This would appear to be difficult, if not impossible, given that the atomic beam in Fig. 1.1 is directed along the y-axis, making it unclear how to align the magnet in the y-direction and measure a deflection. In our depiction and discussion of Stern-Gerlach experiments, we ignore this technical complication.

In the SPINS program, as in real Stern-Gerlach experiments, the numbers of atoms detected in particular states can be predicted by probability rules that we will discuss later. To simplify our schematic depictions of Stern-Gerlach experiments, the numbers shown for detected atoms are those obtained by using the calculated probabilities without any regard to possible statistical uncertainties. That is, if the theoretically predicted probabilities of two measurement possibilities are each 50%, then our schematics will display equal numbers for those two possibilities, whereas in a real experiment, statistical uncertainties might yield a 55%/45% split in one experiment and a 47%/53% split in another, etc. The SPINS program simulations are designed to give statistical uncertainties, so you will need to perform enough experiments to convince yourself that you have a sufficiently good estimate of the probability (see SPINS Lab 1 for more information on statistics).

Now let's consider a series of simple Stern-Gerlach experiments with slight variations that help to illustrate the main features of quantum mechanics. We first describe the experiments and their results and draw some qualitative conclusions about the nature of quantum mechanics. Then we introduce the formal mathematics of the ket notation and show how it can be used to predict the results of each of the experiments.

1.1.1 ■ Experiment 1

The first experiment is shown in Fig. 1.3 and consists of a source of atoms, two Stern-Gerlach analyzers both aligned along the z-axis, and counters for the output ports of the analyzers. The atomic beam coming into the first Stern-Gerlach analyzer is split into two beams at the output, just like the original experiment. Now instead of counting the atoms in the upper output beam, the spin component is measured again by directing those atoms into the second Stern-Gerlach analyzer. The result of this experiment is that no atoms are ever detected coming out of the lower output port of the second Stern-Gerlach analyzer. All atoms that are output from the upper port of the first analyzer also pass

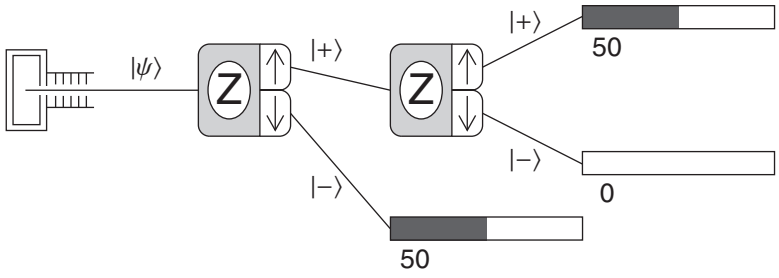

FIGURE 1.3 Experiment 1 measures the spin component along the z-axis twice in succession.

through the upper port of the second analyzer. Thus we say that when the first Stern-Gerlach analyzer measures an atom to have a z-component of spin $S_z = +\hbar/2$, then the second analyzer also measures $S_z = +\hbar/2$ for that atom. This result is not surprising, but it sets the stage for results of experiments to follow.

Though both Stern-Gerlach analyzers in Experiment 1 are identical, they play different roles in this experiment. The first analyzer *prepares* the beam in a particular quantum state ($|+\rangle$) and the second analyzer *measures* the resultant beam, so we often refer to the first analyzer as a **state preparation device**. By preparing the state with the first analyzer, the details of the source of atoms can be ignored. Thus our main focus in Experiment 1 is what happens at the second analyzer because we know that any atom entering the second analyzer is represented by the $|+\rangle$ ket prepared by the first analyzer. All the experiments we will describe employ a first analyzer as a state preparation device, though the SPINS program has a feature where the state of the atoms coming from the oven is determined but unknown, and the user can perform experiments to determine the unknown state using only one analyzer in the experiment.

1.1.2 ■ Experiment 2

The second experiment is shown in Fig. 1.4 and is identical to Experiment 1 except that the second Stern-Gerlach analyzer has been rotated by 90° to be aligned with the x-axis. Now the second analyzer measures the spin component along the x-axis rather the z-axis. Atoms input to the second analyzer are still represented by the ket $|+\rangle$ because the first analyzer is unchanged. The result of this experiment is that atoms appear at both possible output ports of the second analyzer. Atoms leaving the upper port of the second analyzer have been measured to have $S_x = +\hbar/2$, and atoms leaving

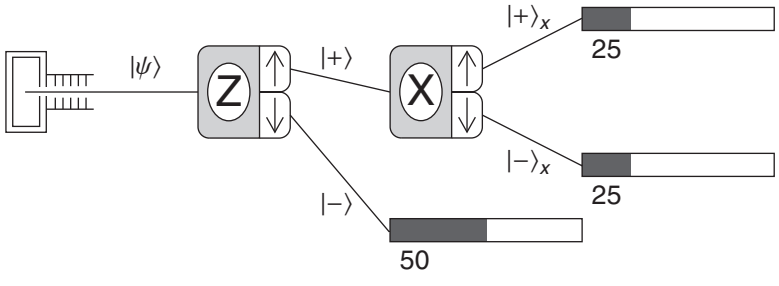

FIGURE 1.4 Experiment 2 measures the spin component along the z-axis and then along the x-axis.

the lower port have $S_x = -\hbar/2$. On average, each of these ports has 50% of the atoms that left the upper port of the first analyzer. As shown in Fig. 1.4, the output states of the second analyzer have new labels $|+\rangle_x$ and $|-\rangle_x$, where the x subscript denotes that the spin component has been measured along the x-axis. We assume that if no subscript is present on the quantum ket (e.g., $|+\rangle$), then the spin component is along the z-axis. This use of the z-axis as the default is a common convention throughout our work and also in much of physics.

A few items are noteworthy about this experiment. First, we notice that there are still only two possible outputs of the second Stern-Gerlach analyzer. The fact that it is aligned along a different axis doesn't affect the fact that we get only two possible results for the case of a spin-1/2 particle. Second, it turns out that the results of this experiment would be unchanged if we used the lower port of the first analyzer. That is, atoms entering the second analyzer in state $|-\rangle$ would also result in half the atoms in each of the $|\pm\rangle_x$ output ports. Finally, we cannot predict which of the second analyzer output ports any particular atom will come out. This can be demonstrated in actual experiments by recording the individual counts out of each port. The arrival sequences at any counter are completely random. We can say only that there is a 50% probability that an atom from the second analyzer will exit the upper analyzer port and a 50% probability that it will exit the lower port. The random arrival of atoms at the detectors can be seen clearly in the SPINS program simulations.

This probabilistic nature is at the heart of quantum mechanics. One might be tempted to say that we just don't know enough about the system to predict which port the atom will exit. That is to say, there may be some other variables, of which we are ignorant, that would allow us to predict the results. Such a viewpoint is known as a **local hidden variable theory**. John Bell proved that such theories are not compatible with the experimental results of quantum mechanics. The conclusion to draw from this is that even though quantum mechanics is a probabilistic theory, it is a complete description of reality. We will have more to say about this in Chapter 4.

Note that the 50% probability referred to above is the probability that an atom input to the second analyzer exits one particular output port. It is not the probability for an atom to pass through the whole system of Stern-Gerlach analyzers. It turns out that the results of this experiment (the 50/50 split at the second analyzer) are the same for any combination of two orthogonal axes of the first and second analyzers.

1.1.3 ■ Experiment 3

Experiment 3, shown in Fig. 1.5, extends Experiment 2 by adding a third Stern-Gerlach analyzer aligned along the z-axis. Atoms entering the third analyzer have been measured by the first Stern-Gerlach analyzer to have spin component up along the z-axis, and by the second analyzer to have spin component up along the x-axis. The third analyzer then measures how many atoms have spin component up or down

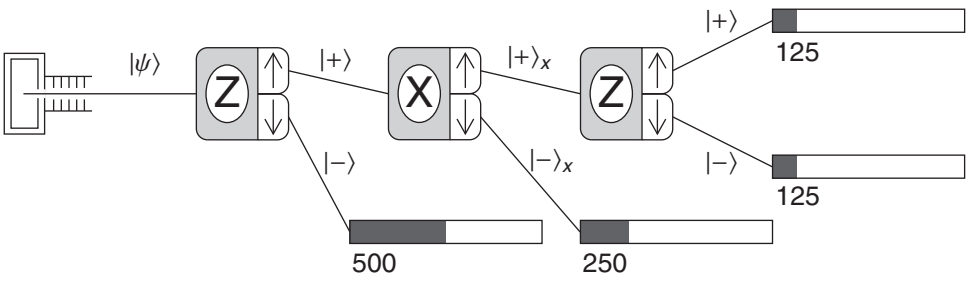

FIGURE 1.5 Experiment 3 measures the spin component three times in succession.

along the z-axis. Classically, one would expect that the final measurement would yield the result spin up along the z-axis, because that was measured at the first analyzer. That is to say: classically the first two analyzers tell us that the atoms have $S_z = +\hbar/2$ and $S_x = +\hbar/2$, so the third measurement must yield $S_z = +\hbar/2$. But that doesn't happen, as Erwin learned with his quantum socks in the Prologue. The quantum mechanical result is that the atoms are split with 50% probability into each output port at the third analyzer. Thus the last two analyzers behave like the two analyzers of Experiment 2 (except with the order reversed), and the fact that there was an initial measurement that yielded $S_z = +\hbar/2$ is somehow forgotten or erased.

This result demonstrates another key feature of quantum mechanics: a measurement disturbs the system. The second analyzer has disturbed the system such that the spin component along the z-axis does not have a unique value, even though we measured it with the first analyzer. Erwin saw this when he sorted, or measured, his socks by color and then by length. When he looked, or measured, a third time, he found that the color he had measured originally was now random—the socks had forgotten about the first measurement. One might ask: Can I be more clever in designing the experiment such that I don't disturb the system? The short answer is no. There is a fundamental incompatibility in trying to measure the spin component of the atom along two different directions. So we say that S_x and S_z are **incompatible observables**. We cannot know the measured values of both simultaneously. The state of the system can be represented by the ket $|+\rangle = |S_z = +\hbar/2\rangle$ or by the ket $|+\rangle_x = |S_x = +\hbar/2\rangle$, but it cannot be represented by a ket $|S_z = +\hbar/2, S_x = +\hbar/2\rangle$ that specifies values of both components. Having said this, it should be said that not all pairs of quantum mechanical observables are incompatible. It is possible to do some experiments without disturbing some of the other aspects of the system. We will see in Section 2.4 that whether two observables are compatible or not is very important in how we analyze a quantum mechanical system.

Not being able to measure both the S_z and S_x spin components is clearly distinct from the classical case where we can measure all three components of the spin vector, which tells us which direction the spin is pointing. In quantum mechanics, the incompatibility of the spin components means that we cannot know which direction the spin is pointing. So when we say "the spin is up," we really mean only that the spin component along that one axis is up (vs. down). The quantum mechanical spin vector cannot be said to be pointing in any given direction. As is often the case, we must check our classical intuition at the door of quantum mechanics.

1.1.4 ■ Experiment 4

Experiment 4 is depicted in Fig. 1.6 and is a slight variation on Experiment 3. Before we get into the details, note a few changes in the schematic drawings. As promised, we have dropped the ket labels on the beams because they are redundant. We have deleted the counters on all but the last analyzer and instead simply blocked the unwanted beams and given the average number of atoms passing from one analyzer to the next. The beam blocks are shown explicitly in Fig. 1.6 but will not be shown after this to be consistent with the SPINS program. Note also that in Experiment 4c two output beams are combined as input to the following analyzer. This is simple in principle and in the SPINS program but can be difficult in practice. The recombination of the beams must be done properly so as to avoid "disturbing" the beams. If you care to read more about this problem, see Feynman's *Lectures on Physics*, volume 3. We will have more to say about the "disturbance" later in Section 2.2. For now we simply assume that the beams can be recombined in the proper manner.

Experiment 4a is identical to Experiment 3. In Experiment 4b, the upper beam of the second analyzer is blocked and the lower beam is sent to the third analyzer. In Experiment 4c, both beams are combined with our new method and sent to the third analyzer. It should be clear from our previous experiments that Experiment 4b has the same results as Experiment 4a. We now ask about the results of

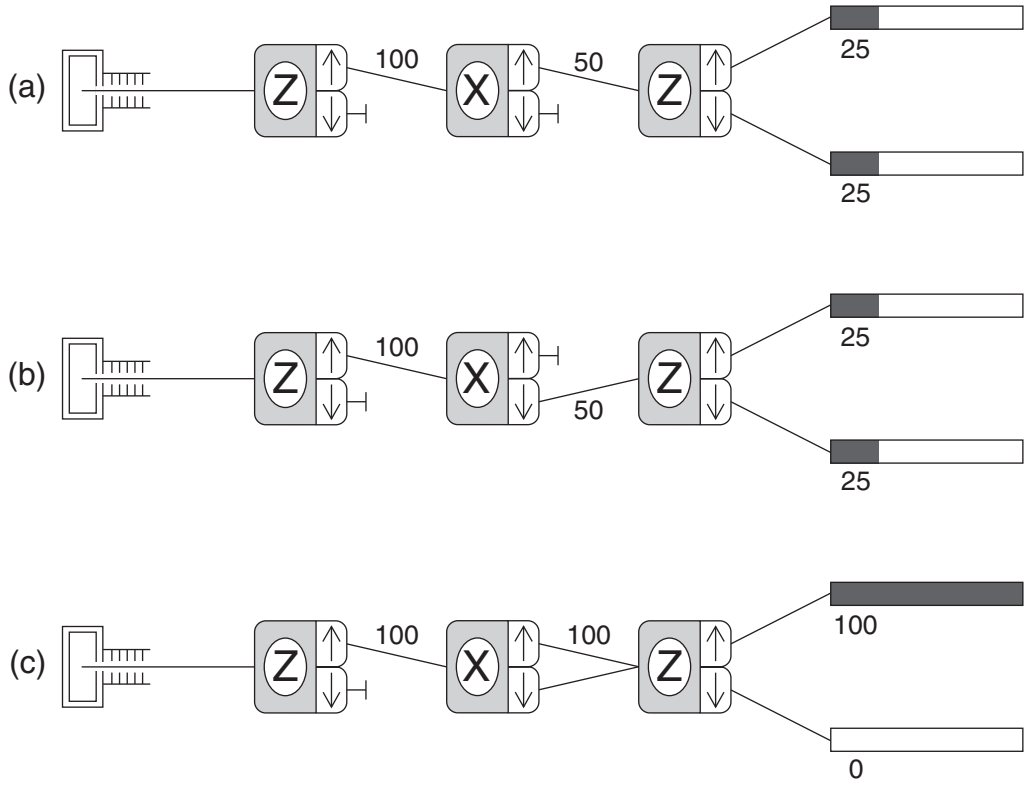

FIGURE 1.6 Experiment 4 measures the spin component three times in succession and uses (a and b) one or (c) two beams from the second analyzer.

Experiment 4c. If we were to use classical probability analysis, then Experiment 4a would indicate that the probability for an atom leaving the first analyzer to take the upper path through the second analyzer and then exit through the upper port of the third analyzer is 25%, where we are now referring to the total probability for those two steps. Likewise, Experiment 4b would indicate that the total probability to take the lower path through the second analyzer and exit through the upper port of the third analyzer is also 25%. Hence the total probability to exit from the upper port of the third analyzer when both paths are available, which is Experiment 4c, would be 50%, and likewise for the exit from the lower port.

However, the quantum mechanical result in Experiment 4c is that all the atoms exit the upper port of the third analyzer and none exits the lower port. The atoms now appear to "remember" that they were initially measured to have spin up along the z-axis. By combining the two beams from the second analyzer, we have avoided the quantum mechanical disturbance that was evident in Experiments 3, 4a, and 4b. The result is now the same as Experiment 1, which means it is as if the second analyzer is not there.

To see how odd this is, look carefully at what happens at the lower port of the third analyzer. In this discussion, we refer to percentages of atoms leaving the first analyzer, because that analyzer is the same in all three experiments. In Experiments 4a and 4b, 50% of the atoms are blocked after the middle analyzer and 25% of the atoms exit the lower port of the third analyzer. In Experiment 4c, 100% of the atoms pass from the second analyzer to the third analyzer, yet fewer atoms come out of the lower port. In fact, no atoms make it through the lower port! So we have a situation where

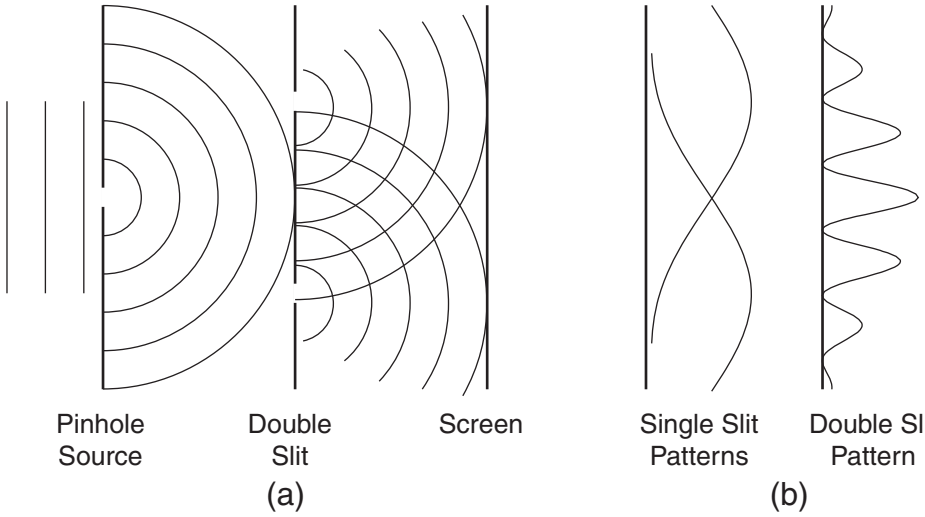

FIGURE 1.7 (a) Young's double-slit interference experiment and (b) resultant intensity patterns observed on the screen, demonstrating single-slit diffraction and double-slit interference.

allowing more ways or paths to reach a counter results in fewer counts. Classical probability theory cannot explain this aspect of quantum mechanics. It is as if you opened a second window in a room to get more sunlight and the room went dark!

However, you may already know of a way to explain this effect. Imagine a procedure whereby combining two effects leads to cancellation rather than enhancement. The concept of wave interference, especially in optics, comes to mind. In the Young's double-slit experiment, light waves pass through two narrow slits and create an interference pattern on a distant screen, as shown in Fig. 1.7. Either slit by itself produces a nearly uniform illumination of the screen, but the two slits combined produce bright and dark interference fringes, as shown in Fig. 1.7(b). We explain this by adding together the electric field vectors of the light from the two slits, then squaring the resultant vector to find the light intensity. We say that we add the amplitudes and then square the total amplitude to find the resultant intensity. See Section 6.6 or an optics textbook for more details about this experiment.

We follow a similar prescription in quantum mechanics. We add together amplitudes and then take the square to find the resultant probability, which opens the door to interference effects. Before we discuss quantum mechanical interference, we must explain what we mean by an amplitude in quantum mechanics and how we calculate it.

1.2 ■ QUANTUM STATE VECTORS

Postulate 1 of quantum mechanics stipulates that kets are to be used for a mathematical description of a quantum mechanical system. These kets are abstract entities that obey many of the rules you know about ordinary spatial vectors. Hence they are called **quantum state vectors**. As we will show in Example 1.3, these vectors must employ complex numbers in order to properly describe quantum mechanical systems. Quantum state vectors are part of a vector space that we call a **Hilbert space**. The dimensionality of the Hilbert space is determined by the physics of the system at hand. In the Stern-Gerlach example, the two possible results for a spin component measurement dictate that the vector space has only two