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978-1-009-30907-3 — Introduction to Probability for Computing
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Part I

Fundamentals and Probability on Events

In this first part of the book we focus on some basic tools that we will need throughout the book.

We start, in Chapter 1, with a review of some mathematical basics: series, limits, integrals, counting, and asymptotic notation. Rather than attempting an exhaustive coverage, we instead focus on a select "toolbox" of techniques and tricks that will come up over and over again in the exercises throughout the book. Thus, while none of this chapter deals with probability, it is worth taking the time to master its contents.

In Chapter 2, we cover the fundamentals of probability. Here we define probability based on an experiment and events. We discuss the axioms of probability, conditioning, independence, the Law of Total Probability, and Bayes' Law.

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1 Before We Start ... Some Mathematical Basics

This book assumes some mathematical skills. The reader should be comfortable with high school algebra, including logarithms. Basic calculus (integration, differentiation, limits, and series evaluation) is also assumed, including nested (3D) integrals and sums. We also assume that the reader is comfortable with sets and with simple combinatorics and counting (as covered in a discrete math class). Finally, we assume versatility with "big-O" and "little-o" notation. To help the reader, in this chapter we review a few basic concepts that come up repeatedly throughout the book. Taking the time to understand these *now* will make it much easier to work through the book.

1.1 Review of Simple Series

There are several series that come up repeatedly in the book, starting in Chapter 3.

Question: Try evaluating the following in closed form. (Don't peek at the answers until you've tried these yourself.) We provide the full derivations below.

(a) $S = 1 + x + x^2 + x^3 + \cdots + x^n$.
(b) $S = 1 + x + x^2 + x^3 + \cdots, \quad$ where $|x| < 1$.
(c) $S = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1}$.
(d) $S = 1 + 2x + 3x^2 + 4x^3 + \cdots, \quad$ where $|x| < 1$.

Example 1.1 *Evaluate:* $S = 1 + x + x^2 + x^3 + \cdots + x^n$.

Solution: The trick here is to multiply both sides by the quantity $(1 - x)$:

$$\begin{aligned}(1-x)S &= S - xS \\ &= 1 + x + x^2 + x^3 + \cdots + x^n \\ &\quad\; - x - x^2 - x^3 - \cdots - x^{n+1} \\ &= 1 - x^{n+1}.\end{aligned}$$

Hence,

$$S = \frac{1 - x^{n+1}}{1 - x}. \tag{1.1}$$

Note that (1.1) assumes that $x \neq 1$. If $x = 1$, then the answer is clearly $S = n + 1$.

Example 1.2 *Evaluate:* $S = 1 + x + x^2 + x^3 + \cdots$ *where* $|x| < 1$.

Solution: This is the same as series (a) except that we need to take the limit as $n \to \infty$:

$$S = \lim_{n\to\infty} 1 + x + x^2 + \cdots + x^n = \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x} = \frac{1}{1 - x}. \tag{1.2}$$

Question: Why did we need $|x| < 1$? What would happen if $|x| \geq 1$?

Answer: If $|x| \geq 1$, then the infinite sum diverges.

Example 1.3 *Evaluate:* $S = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1}$.

Approach 1: One approach is to again use the $(1 - x)$ trick:

$$\begin{aligned}(1 - x)S &= 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} \\ &\quad\; - x - 2x^2 - 3x^3 - 4x^4 - \cdots - nx^n \\ &= 1 + x + x^2 + x^3 + \cdots + x^{n-1} - nx^n \\ &= \frac{1 - x^n}{1 - x} - nx^n \\ &= \frac{1 - (n+1)x^n + nx^{n+1}}{1 - x}.\end{aligned}$$

Hence,

$$S = \frac{1 - (n+1)x^n + nx^{n+1}}{(1 - x)^2}. \tag{1.3}$$

Approach 2: An easier approach is to view the sum as the derivative of a known sum:

$$\begin{aligned}S &= \frac{d}{dx}\left(1 + x + x^2 + x^3 + \cdots + x^n\right) \\ &= \frac{d}{dx}\left(\frac{1 - x^{n+1}}{1 - x}\right) \\ &= \frac{(1 - x)\cdot(-(n+1)x^n) + (1 - x^{n+1})}{(1 - x)^2} \\ &= \frac{1 - (n+1)x^n + nx^{n+1}}{(1 - x)^2}.\end{aligned}$$

The above assumes that $x \neq 1$. If $x = 1$, then the answer is $S = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

Example 1.4 *Evaluate:* $S = 1 + 2x + 3x^2 + 4x^3 + \cdots \quad$ *where* $|x| < 1$.

Solution: We again view S as a derivative of a sum:

$$\begin{aligned} S &= \frac{d}{dx}\left(1 + x + x^2 + x^3 + \cdots\right) \\ &= \frac{d}{dx}\left(\frac{1}{1-x}\right) \\ &= \frac{1}{(1-x)^2}. \end{aligned} \tag{1.4}$$

1.2 Review of Double Integrals and Sums

Integrals, nested integrals, and nested sums come up throughout the book, starting in Chapter 7. When evaluating these, it is important to pay attention to the area over which you're integrating and also to remember tricks like integration by parts.

Question: Try deriving the following three expressions (again, no peeking at the answers).

(a) $\int_0^\infty ye^{-y}dy$.
(b) $\int_0^\infty \int_0^y e^{-y}dxdy$. Do this both with and without changing the order of integration.
(c) $\int_1^e \int_0^{\ln x} 1dydx$. Do this both with and without changing the order of integration.

Below we provide the derivations.

Example 1.5 *Derive:* $\int_0^\infty ye^{-y}dy$.

Solution: We start by reviewing integration by parts:

$$\int_a^b udv = (uv)\Big|_a^b - \int_a^b vdu. \tag{1.5}$$

Applying (1.5), let $u = y$, and $du = dy$. Let $dv = e^{-y}dy$, and $v = -e^{-y}$. Then,

$$\begin{aligned}\int_0^\infty ye^{-y} &= -ye^{-y}\Big|_{y=0}^{y=\infty} - \int_{y=0}^{\infty}(-e^{-y})dy \\ &= 0 - (-0) - e^{-y}\Big|_{y=0}^{\infty} \\ &= 0 + 0 - (0 - 1) \\ &= 1.\end{aligned}$$

Example 1.6 *Derive:* $\int_0^\infty \int_0^y e^{-y}dxdy$.

Solution: Without changing the order of integration, we have:

$$\begin{aligned}\int_{y=0}^{y=\infty}\int_{x=0}^{x=y} e^{-y}dxdy &= \int_{y=0}^{y=\infty} xe^{-y}\Big|_{x=0}^{x=y} dy \\ &= \int_{y=0}^{y=\infty} ye^{-y}dy \\ &= 1.\end{aligned}$$

To change the order of integration, we first need to understand the space over which we're integrating. The original region of integration is drawn in Figure 1.1(a), where y ranges from 0 to ∞, and, for each particular value of y, we let x range from 0 to y.

(a) *Original integration space* (b) *Equivalent integration space*

Figure 1.1 *Region of integration drawn two ways.*

We can visualize this instead as shown in Figure 1.1(b), where x now ranges from 0 to ∞, and, for each particular value of x, we let y range from x to ∞:

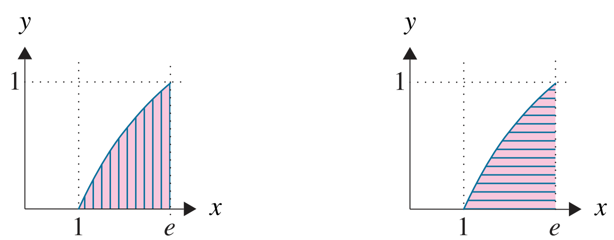
$$\begin{aligned}\int_{x=0}^{x=\infty}\int_{y=x}^{y=\infty} e^{-y}dydx &= \int_{x=0}^{x=\infty} -e^{-y}\Big|_{y=x}^{y=\infty} dx \\ &= \int_{x=0}^{x=\infty}(0 + e^{-x})dx \\ &= -e^{-x}\Big|_{x=0}^{x=\infty} \\ &= 1.\end{aligned}$$

Example 1.7 *Derive:* $\int_1^e \int_0^{\ln x} 1 dy dx$.

Solution: Without changing the order of integration, we have:

$$\begin{aligned}\int_{x=1}^{x=e} \int_{y=0}^{y=\ln x} 1 dy dx &= \int_{x=1}^{x=e} \ln x dx \\ &\quad \text{(applying integration by parts)} \\ &= (\ln x \cdot x)\Big|_{x=1}^{x=e} - \int_{x=1}^{x=e} x \cdot \frac{1}{x} dx \\ &= e - 0 - (e-1) \\ &= 1.\end{aligned}$$

To change the order of integration, we first need to understand the space over which we're integrating. This is drawn in Figure 1.2(a).



(a) Original integration space (b) Equivalent integration space

Figure 1.2 *Region of integration drawn two ways.*

We can visualize this instead as shown in Figure 1.2(b), which leads to the nested integrals:

$$\begin{aligned}\int_{y=0}^{y=1} \int_{x=e^y}^{x=e} 1 dx dy &= \int_{y=0}^{y=1} x\Big|_{x=e^y}^{x=e} dy \\ &= \int_{y=0}^{y=1} (e - e^y) dy \\ &= (ey - e^y)\Big|_{y=0}^{y=1} \\ &= (e-e) - (0 - e^0) \\ &= 1.\end{aligned}$$

1.3 Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus (FTC) will come up in the book starting in Chapter 7. We state it here and provide some intuition for why it holds.

Theorem 1.8 (FTC and extension) *Let $f(t)$ be a continuous function defined on the interval $[a, b]$. Then, for any x, where $a < x < b$,*

$$\frac{d}{dx}\int_a^x f(t)dt = f(x). \tag{1.6}$$

Furthermore, for any differentiable function $g(x)$,

$$\frac{d}{dx}\int_a^{g(x)} f(t)dt = f(g(x)) \cdot g'(x). \tag{1.7}$$

We start with intuition for (1.6):

The integral $\int_a^x f(t)dt$ represents the area under the curve $f(t)$ between $t = a$ and $t = x$. We are interested in the rate at which this area changes for a small change in x.

It helps to think of the integral as a "box" parameterized by x.

$$\text{Box}(x) = \boxed{\int_a^x f(t)dt}.$$

$$\begin{aligned}
\frac{d}{dx}\int_a^x f(t)dt = \frac{d}{dx}\text{Box}(x) &= \lim_{\Delta\to 0}\frac{\text{Box}(x+\Delta) - \text{Box}(x)}{\Delta} \\
&= \lim_{\Delta\to 0}\frac{\int_a^{x+\Delta} f(t)dt - \int_a^x f(t)dt}{\Delta} \\
&= \lim_{\Delta\to 0}\frac{\int_x^{x+\Delta} f(t)dt}{\Delta} \\
&\approx \lim_{\Delta\to 0}\frac{f(x)\cdot \cancel{\Delta}}{\cancel{\Delta}} \qquad f(x) \approx f(x+\Delta) \text{ for tiny } \Delta \\
&= f(x).
\end{aligned}$$

The same argument applies to (1.7):

$$\text{Box}(x) = \boxed{\int_a^{g(x)} f(t)dt}.$$

$$\begin{aligned}
\frac{d}{dx}\int_a^{g(x)} f(t)dt = \frac{d}{dx}\text{Box}(x) &= \lim_{\Delta\to 0}\frac{\text{Box}(x+\Delta)-\text{Box}(x)}{\Delta} \\
&= \lim_{\Delta\to 0}\frac{\int_a^{g(x+\Delta)} f(t)dt - \int_a^{g(x)} f(t)dt}{\Delta} \\
&= \lim_{\Delta\to 0}\frac{\int_{g(x)}^{g(x+\Delta)} f(t)dt}{\Delta} \\
&\approx \lim_{\Delta\to 0}\frac{f(g(x))\cdot(g(x+\Delta)-g(x))}{\Delta} \\
&= f(g(x))\cdot\lim_{\Delta\to 0}\frac{g(x+\Delta)-g(x)}{\Delta} \\
&= f(g(x))\cdot g'(x).
\end{aligned}$$

1.4 Review of Taylor Series and Other Limits

There are certain limits and limiting series which come up repeatedly in this book, so we discuss these here.

Question: What is the famous limit in (1.8) called, and how should we interpret it?

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n. \tag{1.8}$$

Answer: Expression (1.8) is the definition of Euler's number, e, which is an irrational, transcendental number having value approximately 2.7183.

It helps to think about (1.8) in terms of money. Suppose you have m dollars. You are promised a 100% interest rate yearly. If the interest is compounded annually, you will have $2m$ dollars after one year. If the interest is compounded every 6 months, you will have $\left(1+\frac{1}{2}\right)^2 m = \frac{9}{4}m$ dollars after one year. If the interest is compounded every 4 months, you will have $\left(1+\frac{1}{3}\right)^3 m = \frac{64}{27}m$ dollars after one year. Notice how this keeps going up. If the interest is compounded continuously, you will have

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\cdot m = e\cdot m$$

dollars after one year. Big difference!

Question: What, then, is this limit (assume x is a constant):

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \ ?$$

Answer:

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x. \tag{1.9}$$

To see this, let $a = \frac{n}{x}$. As $n \to \infty$, we also have $a \to \infty$:

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{a\to\infty}\left(1+\frac{1}{a}\right)^{ax} = \lim_{a\to\infty}\left(\left(1+\frac{1}{a}\right)^a\right)^x = e^x. \tag{1.10}$$

Question: Let $0 < x < 1$. Let's do some comparisons:

(a) What is bigger, $1 + x$ or e^x?
(b) What is bigger, $1 - x$ or e^{-x}?

Hint: It helps to think about the Taylor series expansion of e^x around $x = 0$.

Answer: For $0 < x < 1$, it turns out that $e^x > 1 + x$ and $e^{-x} > 1 - x$. To see this, we start with a brief reminder of the Taylor series expansion around 0, also known as a Maclaurin series. Consider any function $f(x)$ which is infinitely differentiable at $x = 0$. Let us define

$$p(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots .$$

Observe that the multiplier $\frac{x^n}{n!}$ gets very small for large n. It is easy to see that $p(x)$ is a polynomial that approximates $f(x)$ very well around $x = 0$. In particular, you can see via differentiation that the following are true:

$$\begin{aligned} p(0) &= f(0) \\ p'(0) &= f'(0) \\ p''(0) &= f''(0) \\ p'''(0) &= f'''(0) \\ &\text{etc.} \end{aligned}$$

In fact, Taylor's theorem [71, p.678] says roughly that if x is within the radius of convergence of $p(\cdot)$, then $p(x)$ approaches $f(x)$ as we write out more and more terms of $p(x)$. Expressing $p(x)$ with an infinite number of terms allows us to say that $f(x) = p(x)$.

Returning to our question, we can see that the function $f(x) = e^x$ is infinitely differentiable around 0, and thus, for any x, we can express:

$$e^x = f(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots . \tag{1.11}$$

Thus clearly for any $x > 0$, we have that

$$e^x > 1 + x, \tag{1.12}$$

where $1 + x$ is a very good approximation for e^x when x is very small.

Likewise, we can express $f(x) = e^{-x}$ as

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots . \tag{1.13}$$

Now, when $0 < x < 1$, we see that

$$e^{-x} > 1 - x, \tag{1.14}$$

because $\frac{x^2}{2!} > \frac{x^3}{3!} > \frac{x^4}{4!} > \cdots$. Again, $1 - x$ is a very good approximation for e^{-x} when x is very small.

We end with a discussion of the harmonic series.

Definition 1.9 *The* ***n*****th harmonic number** *is denoted by H_n, where*

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}. \tag{1.15}$$

Example 1.10 (Approximating H_n)

Question: How can we find upper and lower bounds on H_n?

Answer: Figure 1.3 shows the function $f(x) = \frac{1}{x}$ in red. We know how to exactly compute the area under the red curve. Now observe that the area under the red curve is upper-bounded by the sum of the areas in the blue rectangles, which form a harmonic sum. Likewise, the area under the red curve is lower-bounded by the sum of the rectangles with the yellow border, which form a related harmonic sum. Specifically, summing the area in the blue rectangles, we have that:

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \int_1^{n+1} \frac{1}{x}\, dx = \ln(n+1).$$

Likewise, summing the area in the yellow rectangles, we have that:

$$\ln n = \int_1^n \frac{1}{x}\, dx > \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$