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Muriel James
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Chapter 1: Functions

LEARNING INTENTIONS

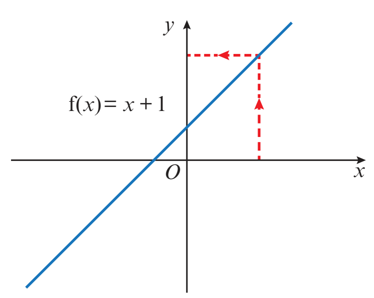
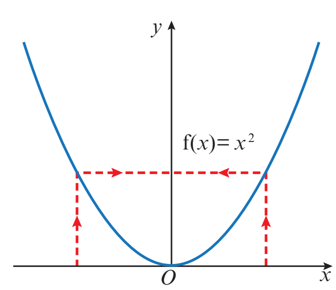
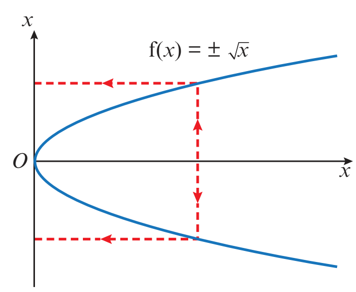
This section will show you how to

- understand and use the terms: function, domain, range (image set), one-one function, inverse function and composition of functions
- use the notation $f(x) = 2x^3 + 5$, $f: x \mapsto 5x - 3$, $f^{-1}(x)$ and $f^2(x)$
- understand the relationship between $y = f(x)$ and $y = |f(x)|$
- solve graphically or algebraically equations of the type $|ax + b| = c$ and $|ax + b| = cx + d$
- explain in words why a given function is a function or why it does not have an inverse
- find the inverse of a one-one function and form composite functions
- sketch graphs to show the relationship between a function and its inverse.

1.1 Mappings

REMINDER

The table below shows one-one, many-one and one-many mappings.

<table>
<tr><th>one-one</th><th>many-one</th><th>one-many</th></tr>
<tr><td>
</td><td>
</td><td>
</td></tr>
<tr><td>For one input value there is just one output value.</td><td>For two input values there is one output value.</td><td>For one input value there are two output values.</td></tr>
</table>

Exercise 1.1

Determine whether each of these mappings is one-one, many-one or one-many.

1 $x \mapsto 2x + 3 \quad x \in \mathbb{R}$

2 $x \mapsto x^2 + 4 \quad x \in \mathbb{R}$

3 $x \mapsto 2x^3 \quad x \in \mathbb{R}$

4 $x \mapsto 3^x \quad x \in \mathbb{R}$

5 $x \mapsto \frac{-1}{x} \quad x \in \mathbb{R}, x > 0$

6 $x \mapsto x^2 + 1 \quad x \in \mathbb{R}, x \geqslant 0$

7 $x \mapsto \frac{2}{x} \quad x \in \mathbb{R}, x > 0$

8 $x \mapsto \pm\sqrt{x} \quad x \in \mathbb{R}, x > 0$

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1.2 Definition of a function

REMINDER

A **function** is a rule that maps each x value to just one y value for a defined set of input values.

This means that mappings that are either $\left\{\begin{array}{l}\text{one-one}\\\text{many-one}\end{array}\right.$ are called functions.

The mapping $x \mapsto x + 1$, where $x \in \mathbb{R}$, is a **one-one function**.

The function can be defined as $\mathrm{f}: x \mapsto x + 1,\ x \in \mathbb{R}$ or $\mathrm{f}(x) = x + 1,\ x \in \mathbb{R}$.

The set of input values for a function is called the **domain** of the function.

The set of output values for a function is called the **range** (or image set) of the function.

WORKED EXAMPLE 1

The function f is defined by $\mathrm{f}(x) = (x - 1)^2 + 4$, for $0 \leqslant x \leqslant 5$.

Find the range of f.

Answers

$\mathrm{f}(x) = (x - 1)^2 + 4$ is a positive quadratic function so the graph will be of the form

$(x - 1)^2 + 4$

This part of the expression is a square so it will always be $\geqslant 0$. The smallest value it can be is 0. This occurs when $x = 1$.

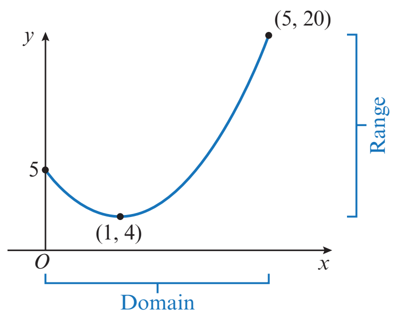
The minimum value of the expression is $0 + 4 = 4$ and this minimum occurs when $x = 1$.

So the function $\mathrm{f}(x) = (x - 1)^2 + 4$ will have a minimum at the point (1, 4).

When $x = 0$, $y = (0 - 1)^2 + 4 = 5$.

When $x = 5$, $y = (5 - 1)^2 + 4 = 20$.

The range is $1 \leqslant \mathrm{f}(x) \leqslant 20$.



Exercise 1.2

1 Which of the mappings in **Exercise 1.1** are functions?

2 Find the range for each of these functions.

a $\mathrm{f}(x) = x - 9,\quad -2 \leqslant x \leqslant 8$

b $\mathrm{f}(x) = 2x - 2,\quad 0 \leqslant x \leqslant 6$

c $\mathrm{f}(x) = 7 - 2x,\quad -3 \leqslant x \leqslant 5$

d $\mathrm{f}(x) = 2x^2,\quad -4 \leqslant x \leqslant 3$

e $\mathrm{f}(x) = 3^x,\quad -4 \leqslant x \leqslant 3$

f $\mathrm{f}(x) = \frac{-1}{x},\quad 1 \leqslant x \leqslant 6$

3 The function g is defined as $g(x) = x^2 - 5$ for $x \geqslant 0$.
Find the range of g.

4 The function f is defined by $f(x) = 4 - x^2$ for $x \in \mathbb{R}$.
Find the range of f.

5 The function f is defined by $f(x) = 3 - (x - 1)^2$ for $x \geqslant 1$.
Find the range of f.

6 The function f is defined by $f(x) = (4x + 1)^2 - 2$ for $x \geqslant -\frac{1}{4}$.
Find the range of f.

7 The function f is defined by $f : x \mapsto 8 - (x - 3)^2$ for $2 \leqslant x \leqslant 7$.
Find the range of f.

8 The function f is defined by $f(x) = 3 - \sqrt{x - 1}$ for $x \geqslant 1$.
Find the range of f.

9 Find the largest possible domain for the following functions.

a $f(x) = \dfrac{1}{x + 3}$ **b** $f(x) = \dfrac{3}{x - 2}$ **c** $f(x) = \dfrac{4}{(x - 3)(x + 2)}$

d $f(x) = \dfrac{1}{x^2 - 4}$ **e** $f : x \mapsto \sqrt{x^3 - 4}$ **f** $f : x \mapsto \sqrt{x + 5}$

g $g : x \mapsto \dfrac{1}{\sqrt{x - 2}}$ **h** $f : x \mapsto \dfrac{x}{\sqrt{3 - 3x}}$ **i** $f : x \mapsto 1 - x^2$

1.3 Composite functions

REMINDER

- When one function is followed by another function, the resulting function is called a **composite function.**
- $fg(x)$ means the function g acts on x first, then f acts on the result.
- $f^2(x)$ means $ff(x)$, so you apply the function f twice.

WORKED EXAMPLE 2

$f : x \mapsto 4x + 3$, for $x \in \mathbb{R}$
$g : x \mapsto 2x^2 - 5$, for $x \in \mathbb{R}$
Find $fg(3)$.

Answers

$fg(3) = f(2 \times 3^2 - 5)$ g acts on 3 first and $g(3) = 2 \times 3^2 - 5 = 13$.
$= f(13)$
$= 4 \times 13 + 3$
$= 55$

WORKED EXAMPLE 3

$g(x) = 2x^2 - 2$, for $x \in \mathbb{R}$
$h(x) = 4 - 3x$, for $x \in \mathbb{R}$
Solve the equation $hg(x) = -14$.

Answers

$hg(x) = h(2x^2 - 2)$	g acts on x first and $g(x) = 2x^2 - 2$.
$= 4 - 3(2x^2 - 2)$	h is the function 'triple and take from 4'
$= 4 - 6x^2 + 6$	expand the brackets
$= 10 - 6x^2$	
$hg(x) = -14$	
$-14 = 10 - 6x^2$	set up and solve the equation
$24 = 6x^2$	
$4 = x^2$	
$x = \pm 2$	

Exercise 1.3

1 $f(x) = 2 - x^2$, for $x \in \mathbb{R}$
$g(x) = \frac{x}{2} + 3$, for $x \in \mathbb{R}$
Find the value of $gf(4)$.

2 $f(x) = (x - 2)^2 - 2$, for $x \in \mathbb{R}$
Find $f^2(3)$.

3 The function f is defined by $f(x) = 1 + \sqrt{x - 3}$, for $x \geqslant 3$.
The function g is defined by $g(x) = \frac{-3}{x} - 1$, for $x > 0$.
Find $gf(7)$.

4 The function f is defined by $f(x) = (x - 2)^2 + 3$, for $x > -2$.
The function g is defined by $g(x) = \frac{3x + 4}{x + 2}$, for $x > 2$.
Find $fg(6)$.

5 $f : x \mapsto 3x - 1$, for $x > 0$
$g : x \mapsto \sqrt{x}$, for $x > 0$
Express each of the following in terms of f and g.

a $x \mapsto 3\sqrt{x} - 1$ **b** $x \mapsto \sqrt{3x - 1}$

6 The function f is defined by $f : x \mapsto 2x - 1$, for $x \in \mathbb{R}$.
The function g is defined by $g : x \mapsto \frac{8}{4 - x}$, for $x \neq 4$.
Solve the equation $gf(x) = 5$.

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7 $f(x) = 2x^2 + 3$, for $x > 0$

$g(x) = \frac{5}{x}$, for $x > 0$

Solve the equation $fg(x) = 4$.

8 The function f is defined by $f : x \mapsto \frac{2x - 1}{x - 3}$, for $x \in \mathbb{R}$, $x \neq 3$.

The function g is defined by $g : x \mapsto \frac{x + 1}{2}$, for $x \in \mathbb{R}$, $x \neq 1$.

Solve the equation $fg(x) = 4$.

9 The function g is defined by $g(x) = 1 - 2x^2$ for $x \geqslant 0$.
The function h is defined by $h(x) = 3x - 1$ for $x \geqslant 0$.
Solve the equation $gh(x) = -3$, giving your answer(s) as exact value(s).

10 The function f is defined by $f : x \mapsto x^2$, for $x \in \mathbb{R}$.
The function g is defined by $g : x \mapsto x + 2$, for $x \in \mathbb{R}$.
Express each of the following as a composite function, using only f and g.

a $x \mapsto (x + 2)^2$ **b** $x \mapsto x^2 + 2$ **c** $x \mapsto x + 4$ **d** $x \mapsto x^4$

11 The functions f and g are defined by $f : x \mapsto x + 3$ and $g : x \mapsto \sqrt{x}$, for $x > 0$.
Express in terms of f and g.

a $x \mapsto \sqrt{x + 3}$ **b** $x \mapsto x + 6$ **c** $x \mapsto \sqrt{x} + 3$

12 Functions f and g are defined as $f(x) = \sqrt{x}$ and $g(x) = \frac{x - 5}{2x + 1}$.

a Find the largest possible domain of g and the corresponding range.

b Solve the equation $g(x) = 0$.

c Find the domain and range of fg.

TIP

Before writing your final answers, compare your solutions with the domains of the original functions.

1.4 Modulus functions

REMINDER

- The **modulus** (or **absolute value**) of a number is the magnitude of the number without a sign attached.
- The modulus of x, written as $|x|$, is defined as

$$|x| = \begin{cases} x & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

- The statement $|x| = k$, where $k \geqslant 0$, means that $x = k$ or $x = -k$.

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WORKED EXAMPLE 4

Solve.

a $|4x + 3| = x + 18$ **b** $|2x^2 - 7| = 9$

Answers

a $|4x + 3| = x + 18$

$4x + 3 = x + 18$ or $4x + 3 = -x - 18$

$3x = 15$ $\quad$ $5x = -21$

$x = 5$ $\quad$ $x = -\frac{21}{5}$

Solution is: $x = 5$ or $-\frac{21}{5}$

b $|2x^2 - 7| = 9$

$2x^2 - 7 = 9$ or $2x^2 - 7 = -9$

$2x^2 = 16$ $\quad$ $2x^2 = -2$

$x^2 = 8$ $\quad$ $x^2 = -1$ no real solution

$x = \pm 2\sqrt{2}$

Solution is: $x = \pm 2\sqrt{2}$

Exercise 1.4

1 Solve.

a $|2x - 1| = 11$ **b** $|2x + 4| = 8$ **c** $|6 - 3x| = 4$

d $\left|\frac{x-2}{5}\right| = 6$ **e** $\left|\frac{3x+4}{3}\right| = 4$ **f** $\left|\frac{9-2x}{3}\right| = 4$

g $\left|\frac{x}{3} - 6\right| = 1$ **h** $\left|\frac{2x+5}{3} + \frac{2x}{5}\right| = 3$ **i** $|2x - 6| = x$

2 Solve.

a $\left|\frac{2x-5}{x+4}\right| = 3$ **b** $\left|\frac{4x+2}{x+3}\right| = 3$ **c** $\left|1 + \frac{2x+5}{x+3}\right| = 4$

d $|2x - 3| = 3x$ **e** $2x + |3x - 4| = 5$ **f** $7 - |1 - 2x| = 3x$

3 Solve, giving your answers as exact values if appropriate.

a $|x^2 - 4| = 5$ **b** $|x^2 + 5| = 11$ **c** $|9 - x^2| = 3 - x$

d $|x^2 - 3x| = 2x$ **e** $|x^2 - 16| = 2x + 1$ **f** $|2x^2 - 1| = x + 2$

g $|3 - 2x^2| = x$ **h** $|x^2 - 4x| = 3 - 2x$ **i** $|2x^2 - 2x + 5| = 1 - x$

4 Solve each pair of simultaneous equations.

a $y = x + 4$
$y = |x^2 - 2|$

b $y = 1 - x$
$y = |4x^2 - 4x|$

TIP

Remember to check your answers to make sure that they satisfy the original equation.

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1.5 Graphs of $y = |f(x)|$ where $f(x)$ is linear

Exercise 1.5

1 Sketch the graphs of each of the following functions, showing the coordinates of the points where the graph meets the axes.

a $y = |x - 2|$ b $y = |3x - 3|$ c $y = |3 - x|$

d $y = \left|\frac{1}{3}x - 3\right|$ e $y = |6 - 3x|$ f $y = \left|5 - \frac{1}{2}x\right|$

2 a Complete the table of values for $y = 3 - |x - 1|$.

x	−2	−1	0	1	2	3	4
y		1		3			

b Draw the graph of $y = 3 - |x - 1|$, for $-2 \leqslant x \leqslant 4$.

3 Draw the graphs of each of the following functions.

a $y = |2x| + 2$ b $y = |x| - 2$ c $y = 4 - |3x|$

d $y = |x - 1| + 3$ e $y = |3x - 6| - 2$ f $y = 4 - \left|\frac{1}{2}x\right|$

4 Given that each of these functions is defined for the domain $-3 \leqslant x \leqslant 4$, find the range of

a $f : x \mapsto 6 - 3x$ b $g : x \mapsto |6 - 3x|$ c $h : x \mapsto 6 - |3x|$

5 Find the range of each function for $-1 \leqslant x \leqslant 5$.

a $f : x \mapsto 2 - 2x$ b $g : x \mapsto |2 - 2x|$ c $h : x \mapsto 2 - |2x|$

6 a Sketch the graph of $y = |3x - 2|$ for $-4 < x < 4$, showing the coordinates of the points where the graph meets the axes.

b On the same diagram, sketch the graph of $y = x + 3$.

c Solve the equation $|3x - 2| = x + 3$.

7 A function f is defined by $f(x) = 2 - |3x - 1|$, for $-1 \leqslant x \leqslant 3$.

a Sketch the graph of $y = f(x)$.

b State the range of f.

c Solve the equation $f(x) = -2$.

8 a On a single diagram, sketch the graphs of $x + 3y = 6$ and $y = |x + 2|$.

b Solve the inequality $|x + 2| < \frac{1}{3}(6 - x)$.

1.6 Inverse functions

REMINDER

- The inverse of the function $f(x)$ is written as $f^{-1}(x)$.
- The domain of $f^{-1}(x)$ is the range of $f(x)$.
- The range of $f^{-1}(x)$ is the domain of $f(x)$.
- It is important to remember that not every function has an inverse.
- An inverse function $f^{-1}(x)$ can exist if, and only if, the function $f(x)$ is a one-one mapping.

WORKED EXAMPLE 5

$f(x) = (x + 3)^2 - 1$, for $x > -3$

a Find an expression for $f^{-1}(x)$.

b Solve the equation $f^{-1}(x) = 3$.

Answers

a $f(x) = (x + 3)^2 - 1$, for $x > -3$

St	Write the function as $y = \ldots$	$y = (x + 3)^2 - 1$
St	Interchange the x and y variables.	$x = (y + 3)^2 - 1$
St	Rearrange to make y the subject.	$x + 1 = (y + 3)^2$
		$\sqrt{x + 1} = y + 3$
		$y = \sqrt{x + 1} - 3$

$f^{-1}(x) = \sqrt{x + 1} - 3$

b $f^{-1}(x) = 3$.

$\sqrt{x + 1} - 3 = 3$

$\sqrt{x + 1} = 6$

$x + 1 = 36$

$x = 35$

Exercise 1.6

1 $f(x) = (x + 2)^2 - 3$, for $x \geqslant -2$.
Find an expression for $f^{-1}(x)$.

2 $f(x) = \dfrac{5}{x - 2}$, for $x \geqslant 0$.
Find an expression for $f^{-1}(x)$.

3 $f(x) = (3x - 2)^2 + 3$, for $x \geqslant \dfrac{2}{3}$.
Find an expression for $f^{-1}(x)$.

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4 $f(x) = 4 - \sqrt{x - 2}$, for $x \geqslant 2$.
Find an expression for $f^{-1}(x)$.

5 $f : x \mapsto 3x - 4$, for $x > 0$.
$g : x \mapsto \frac{4}{4 - x}$, for $x \neq 4$.
Express $f^{-1}(x)$ and $g^{-1}(x)$ in terms of x.

6 $f(x) = (x - 2)^2 + 3$, for $x > 2$.
a Find an expression for $f^{-1}(x)$.
b Solve the equation $f^{-1}(x) = f(4)$.

7 $g(x) = \frac{3x + 1}{x - 3}$, for $x > 3$
a Find an expression for $g^{-1}(x)$ and comment on your result.
b Solve the equation $g^{-1}(x) = 6$.

8 $f(x) = \frac{x}{2} - 2$, for $x \in \mathbb{R}$ $g(x) = x^2 - 4x$, for $x \in \mathbb{R}$
a Find $f^{-1}(x)$.
b Solve $fg(x) = f^{-1}(x)$, leaving answers as exact values.

9 $f : x \mapsto \frac{3x + 1}{x - 1}$, for $x \neq 1$ $g : x \mapsto \frac{x - 2}{3}$, for $x > -2$
Solve the equation $f(x) = g^{-1}(x)$.

10 If $f(x) = \frac{x^2 - 9}{x^2 + 4}$, $x \in \mathbb{R}$, find an expression for $f^{-1}(x)$.

11 If $f(x) = 2\sqrt{x}$ and $g(x) = 5x$, solve the equation $f^{-1}g(x) = 0.01$.

12 Find the value of the constant k such that $f(x) = \frac{2x - 4}{x + k}$ is a self-inverse function.

TIP

A **self-inverse function** is one for which $f(x) = f^{-1}(x)$, for all values of x in the domain.

13 The function f is defined by $f(x) = x^3$. Find an expression for $g(x)$ in terms of x for each of the following.
a $fg(x) = 3x + 2$
b $gf(x) = 3x + 2$

14 Given that $f(x) = 2x + 1$ and $g(x) = \frac{x + 1}{2}$, find the following.
a f^{-1} **b** g^{-1} **c** $(fg)^{-1}$ **d** $(gf)^{-1}$ **e** $f^{-1}g^{-1}$ **f** $g^{-1}f^{-1}$
Write down any observations from your results.

15 Given that $fg(x) = \frac{x + 2}{3}$ and $g(x) = 2x + 5$, find $f(x)$.

16 Functions f and g are defined for all real numbers.
$g(x) = x^2 + 7$ and $gf(x) = 9x^2 + 6x + 8$.
Find $f(x)$.

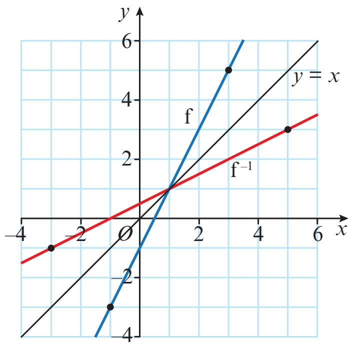
1.7 The graph of a function and its inverse

REMINDER

The graphs of f and f^{-1} are reflections of each other in the line $y = x$.

This is true for all one-one functions and their inverse functions.

This is because: $ff^{-1}(x) = x = f^{-1}f(x)$



Some functions are called **self-inverse functions** because f and its inverse f^{-1} are the same.

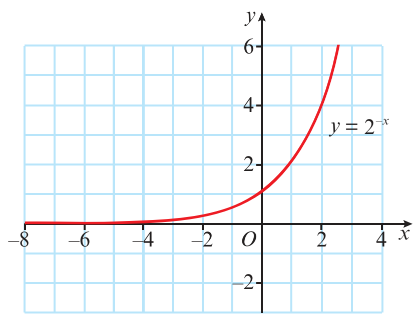
If $f(x) = \frac{1}{x}$, for $x \neq 0$, then $f^{-1}(x) = \frac{1}{x}$, for $x \neq 0$.

So $f(x) = \frac{1}{x}$, for $x \neq 0$, is an example of a self-inverse function.

When a function f is self-inverse, the graph of f will be symmetrical about the line $y = x$.

Exercise 1.7

1 On a copy of the grid, draw the graph of the inverse of the function $y = 2^{-x}$.



2 $f(x) = x^2 + 5, x \geqslant 0$.

On the same axes, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, showing the coordinates of any points where the curves meet the coordinate axes.

3 $g(x) = \frac{1}{2}x^2 - 4$, for $x \geqslant 0$.

Sketch, on a single diagram, the graphs of $y = g(x)$ and $y = g^{-1}(x)$, showing the coordinates of any points where the curves meet the coordinate axes.