

Contents

<i>Preface</i>	<i>page</i> xv
<i>Acknowledgements</i>	xviii
<i>List of Symbols</i>	xx
1 Introduction	1
1.1 On the Einstein field equations	1
1.2 Exact solutions	3
1.3 The Cauchy problem in general relativity	4
1.4 Conformal geometry and general relativity	8
1.5 Existence of asymptotically simple spacetimes	15
1.6 Perspectives	21
1.7 Structure of this book	21
Part I Geometric tools	25
2 Differential geometry	27
2.1 Manifolds	27
2.2 Vectors and tensors on a manifold	30
2.3 Maps between manifolds	36
2.4 Connections, torsion and curvature	38
2.5 Metric tensors	44
2.6 Frame formalisms	51
2.7 Congruences and submanifolds	53
2.8 Further reading	63
3 Spacetime spinors	64
3.1 Algebra of 2-spinors	65
3.2 Calculus of spacetime spinors	81
3.3 Global considerations	90
3.4 Further reading	91
Appendix: the Newman-Penrose formalism	91
4 Space spinors	94
4.1 Hermitian inner products and 2-spinors	94
4.2 The space spinor formalism	97
4.3 Calculus of space spinors	105
4.4 Further reading	111

5	Conformal geometry	112
5.1	Basic concepts of conformal geometry	112
5.2	Conformal transformation formulae	114
5.3	Weyl connections	119
5.4	Spinorial expressions	122
5.5	Conformal geodesics	126
5.6	Further reading	137
Part II General relativity and conformal geometry		139
6	Conformal extensions of exact solutions	141
6.1	Preliminaries	141
6.2	The Minkowski spacetime	145
6.3	The de Sitter spacetime	155
6.4	The anti-de Sitter spacetime	159
6.5	Conformal extensions of static and stationary black hole spacetimes	163
6.6	Further reading	177
7	Asymptotic simplicity	178
7.1	Basic definitions	178
7.2	Other related definitions	181
7.3	Penrose's proposal	182
7.4	Further reading	183
8	The conformal Einstein field equations	184
8.1	A singular equation for the conformal metric	184
8.2	The metric regular conformal field equations	185
8.3	Frame and spinorial formulation of the conformal field equations	194
8.4	The extended conformal Einstein field equations	200
8.5	Further reading	209
9	Matter models	211
9.1	General properties of the conformal treatment of matter models	211
9.2	The Maxwell field	213
9.3	The scalar field	216
9.4	Perfect fluids	219
9.5	Further reading	221
10	Asymptotics	222
10.1	Basic set up: general structure of the conformal boundary	223
10.2	Peeling properties	226
10.3	The Newman-Penrose gauge	231
10.4	Other aspects of asymptotics	238
10.5	Further reading	241
Appendix: spin-weighted functions		241

Contents

xi

Part III Methods of the theory of partial differential equations	245
11 The conformal constraint equations	247
11.1 General setting and basic formulae	247
11.2 Basic notions of elliptic equations	252
11.3 The Hamiltonian and momentum constraints	253
11.4 The conformal constraint equations	259
11.5 The constraints on compact manifolds	268
11.6 Asymptotically Euclidean manifolds	270
11.7 Hyperboloidal manifolds	284
11.8 Other methods for solving the constraint equations	286
11.9 Further reading	291
Appendix: some results of analysis	291
12 Methods of the theory of hyperbolic differential equations	294
12.1 Basic notions	294
12.2 Uniqueness and domains of dependence	301
12.3 Local existence results for symmetric hyperbolic systems	306
12.4 Local existence for boundary value problems	313
12.5 Local existence for characteristic initial value problems	319
12.6 Concluding remarks	328
12.7 Further reading	329
Appendix	329
13 Hyperbolic reductions	331
13.1 A model problem: the Maxwell equations on a fixed background	332
13.2 Hyperbolic reductions using gauge source functions	336
13.3 The subsidiary equations for the standard conformal field equations	354
13.4 Hyperbolic reductions using conformal Gaussian systems	366
13.5 Other hyperbolic reduction strategies	383
13.6 Further reading	385
Appendix A.1: the reduced Einstein field equations	386
Appendix A.2: differential forms	389
14 Causality and the Cauchy problem in general relativity	390
14.1 Basic elements of Lorentzian causality	390
14.2 PDE causality versus Lorentzian causality	395
14.3 Cauchy developments and maximal Cauchy developments	398
14.4 Stability of solutions	401
14.5 Causality and conformal geometry	402
14.6 Further reading	404

Part IV Applications	405
15 De Sitter-like spacetimes	407
15.1 The de Sitter spacetime as a solution to the conformal field equations	408
15.2 Perturbations of initial data for the de Sitter spacetime	416
15.3 Global existence and stability using gauge source functions	420
15.4 Global existence and stability using conformal Gaussian systems	426
15.5 Extensions	434
15.6 Further reading	436
16 Minkowski-like spacetimes	437
16.1 The Minkowski spacetime and the conformal field equations	438
16.2 Perturbations of hyperboloidal data for the Minkowski spacetime	442
16.3 A priori structure of the conformal boundary	444
16.4 The proof of the main existence and stability result	450
16.5 Extensions and further reading	452
17 Anti-de Sitter-like spacetimes	454
17.1 General properties of anti-de Sitter-like spacetimes	455
17.2 The formulation of an initial boundary value problem	460
17.3 Covariant formulation of the boundary conditions	468
17.4 Other approaches to the construction of anti-de Sitter-like spacetimes	475
17.5 Further reading	475
18 Characteristic problems for the conformal field equations	477
18.1 Geometric and gauge aspects of the standard characteristic initial value problem	478
18.2 The conformal evolution equations in the standard characteristic initial value problem	485
18.3 A local existence result for characteristic problems	491
18.4 The asymptotic characteristic problem on a cone	496
18.5 Further reading	502
19 Static solutions	504
19.1 The static field equations	504
19.2 Analyticity at infinity	512
19.3 A regularity condition	517
19.4 Multipole moments	518
19.5 Uniqueness of the conformal structure of static metrics	522
19.6 Characterisation of static initial data	524
19.7 Further reading	525
Appendix 1: Hölder conditions	526
Appendix 2: the Cauchy-Kowalewskaya theorem	526

<i>Contents</i>	xiii
20 Spatial infinity	527
20.1 Cauchy data for the conformal field equations near spatial infinity	527
20.2 Massless and purely radiative spacetimes	531
20.3 A regular initial value problem at spatial infinity	538
20.4 Spatial infinity and peeling	554
20.5 Existence of asymptotically simple spacetimes	555
20.6 Obstructions to the smoothness of null infinity	556
20.7 Further reading	557
Appendix: properties of functions on the complex null cone	558
21 Perspectives	560
21.1 Stability of cosmological models	560
21.2 Stability of black hole spacetimes	562
21.3 Conformal methods and numerics	564
21.4 Computer algebra	567
21.5 Concluding remarks	568
<i>References</i>	569
<i>Index</i>	587