1
Introduction

This book discusses an approach to the analysis of asymptotic and global properties of solutions to the equations of Einstein's theory of general relativity (the Einstein field equations) based on ideas arising in conformal geometry. This approach allows a geometric and rigorous formulation of problems and notions of great physical relevance in the context of general relativity. At the same time, it provides valuable insights into the properties of the Einstein field equations under optimal regularity conditions.

Before entering into the subject, it is useful to discuss the motivation behind this type of endeavour. Accordingly, a brief account of certain aspects of what can be called *mathematical general relativity* is necessary.

1.1 On the Einstein field equations

Einstein's theory of general relativity is the best theory of gravity we have. It is a relativistic theory of gravity which considers four-dimensional differentiable, orientable manifolds $\tilde{\mathcal{M}}$ endowed with a Lorentzian metric $\tilde{\boldsymbol{g}}$; a discussion of these differential geometric notions is provided in Chapter 2. The pair $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ is called a ***spacetime***. Here, and in the rest of this book, quantities associated to the spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ will be distinguished by a tilde (˜); the motivation behind this notation will become clear in the following. The gravitational field is described in general relativity as a manifestation of the curvature of spacetime.

The fundamental equations of general relativity, the ***Einstein field equations***, describe how matter produces the curvature of spacetime. They are given, in the abstract index notation discussed in Section 2.2.6, by

$$\tilde{R}_{ab} - \frac{1}{2}\tilde{R}\tilde{g}_{ab} + \lambda\tilde{g}_{ab} = \tilde{T}_{ab}, \tag{1.1}$$

where $\tilde{g}_{ab}$ is the abstract index version of $\tilde{\boldsymbol{g}}$, and where $\tilde{R}_{ab}$ and $\tilde{R}$ denote, respectively, the Ricci tensor and Ricci scalar of the metric $\tilde{\boldsymbol{g}}$. Moreover, λ is the so-called ***cosmological constant*** and $\tilde{T}_{ab}$ denotes the energy–momentum tensor of the matter in the spacetime. Precise definitions and conventions for the curvature tensors are provided in Chapter 2, while a discussion of the energy–momentum tensors for a range of matter models is provided in Chapter 9. The energy–momentum tensor satisfies the ***conservation equation***

$$\tilde{\nabla}^a \tilde{T}_{ab} = 0,$$

where $\tilde{\nabla}_a$ denotes the covariant derivative of the metric $\tilde{\boldsymbol{g}}$. The ***Bianchi identity*** satisfied by the Riemann curvature tensor $\tilde{R}^a{}_{bcd}$ of the metric $\tilde{\boldsymbol{g}}$ ensures the consistency between the conservation equation and the Einstein field equations. A ***solution to the Einstein field equations*** is a pair $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$, together with a $\tilde{\boldsymbol{g}}$-divergence-free tensor $\tilde{T}_{ab}$ such that Equation (1.1) holds. In suitable open subsets of $\tilde{\mathcal{M}}$ the metric $\tilde{\boldsymbol{g}}$ is expressed, using some *local coordinates* (x^μ), in terms of its components $(\tilde{g}_{\mu\nu})$; here and in what follows, Greek indices are used as *coordinate indices*. In general, several coordinate charts will be needed to cover the spacetime manifold $\tilde{\mathcal{M}}$. Two metrics $\tilde{\boldsymbol{g}}$ and $\bar{\boldsymbol{g}}$ over $\tilde{\mathcal{M}}$ are said to be ***isometric*** if they are related, everywhere on $\tilde{\mathcal{M}}$, by some coordinate transformation.

In the cases where $\tilde{T}_{ab} = 0$, a direct computation shows that Equation (1.1) implies

$$\tilde{R}_{ab} = \lambda \tilde{g}_{ab}. \tag{1.2}$$

In what follows, the latter will be known as the ***vacuum Einstein field equations*** and a solution thereof as an ***Einstein spacetime***. The full curvature of a four-dimensional manifold is described by the tensor $\tilde{R}^a{}_{bcd}$. This tensor has 20 independent components. By contrast, the Ricci tensor appearing in the Einstein field Equations (1.1) and (1.2) has only 10 independent components. Hence, even in the absence of a cosmological constant, where the vacuum field Equations (1.2) reduce to

$$\tilde{R}_{ab} = 0, \tag{1.3}$$

it is possible to have solutions with a non-vanishing Riemann tensor. As a consequence, solutions to the vacuum field equations play a special role in general relativity, as they describe *pure gravitational configurations*. Vacuum spacetimes are often deemed more fundamental, as they exclude potential pathologies which may arise from the choice of a particular matter model.

General relativity has two main domains of applicability: *cosmology* and *isolated systems*. To make use of the Einstein field Equations (1.1) within these two domains, one requires a number of idealisations. On the one hand, in cosmology it is usually assumed that the matter content of the universe can be described by a perfect fluid with an equation of state which depends on a particular cosmological era. It is a convention in mathematical relativity to refer to spacetimes with compact spacelike sections as ***cosmological spacetimes***. On the other hand, ***isolated systems*** are convenient idealisations of astrophysical objects for which it is assumed that the cosmological expansion has no influence. The transition between the regime of isolated systems and the cosmological one is a topic of fundamental relevance for the understanding of the physical content of the Einstein field equations; see, for example, Ellis (1984, 2002).

The validity of general relativity has been verified in a number of experiments covering a wide range of scenarios ranging from the dynamics of the solar system to cosmological scales; see, for example, Will (2014) for a discussion of the subject. Surveys of the physical content of general relativity and its various domains of applicability can be found, for example, in Poisson and Will (2015) and Shapiro (1999).

Note. *In the remainder of this chapter, in order to simplify the presentation, the discussion will be restricted to Einstein spaces, that is, solutions to the vacuum Equations* (1.2). *The inclusion of matter very often requires a case-by-case analysis.*

1.2 Exact solutions

A natural first step to developing an understanding of the properties of solutions to the Einstein field equations is the construction of ***exact solutions***, that is, explicit solutions written in terms of *elementary functions* of some coordinates. The first non-trivial exact solution to the Einstein field equations ever obtained is the Schwarzschild solution. It describes a static spherically symmetric vacuum configuration; see Schwarzschild (1916), an English translation of which can be found in Schwarzschild (2003). Remarkably, despite the complexity of the field equations, the literature contains a vast number of exact solutions to the equations of general relativity; see, for example, Stephani et al. (2003) for a monograph on the subject. The number of solutions with a physical or geometric significance is, arguably, much smaller; see, for example, Bičák (2000) and Griffiths and Podolský (2009).

1.2.1 Construction of exact solutions

The construction of exact solutions to the Einstein field equations requires a number of assumptions concerning the nature of the solutions. The most natural assumptions involve the presence of continuous symmetries (*Killing vectors*) of some type in the solution, for example, spherical symmetry, axial symmetry, stationarity (including staticity) and homogeneity. Other types of assumptions involve the algebraic structure of the curvature tensors of the spacetime (e.g. the Petrov type of the Weyl tensor). These types of assumptions are harder to justify on a physical basis.

Exact solutions are usually constructed in a coordinate system adapted to the assumptions being made. Very often, these *natural coordinates* cover only a portion of the whole spacetime manifold. Thus, one needs to find new coordinate systems (charts) for the exact solution which allow one to uncover a full *maximal* *analytic extension* of the spacetime. This maximal extension usually paves the way to the interpretation of the exact solution and gives access to its global properties.

1.2.2 The limitations of exact solutions

Several of the well-known consequences of general relativity have been developed through the analysis of exact solutions, for example, the notion of a black hole. Thus, the study of exact solutions to the Einstein field equations helps to develop a physical and geometric intuition which, in turn, can lead to questions concerning more generic solutions. However, despite the valuable insights they provide, the construction of exact solutions is not a systematic approach to explore the *space of solutions of the theory*. In particular, this approach leaves open the question of whether certain properties of a solution are *generic*, that is, satisfied by a broader class of spacetimes. Moreover, exact solutions do not lend themselves to the analysis of dynamic situations such as, for example, the description of the gravitational radiation produced by an isolated system. Thus, it is not possible to address issues involving *stability* just by means of exact solutions. In order to analyse the above issues one has to consider whether it is possible to formulate an *initial value problem for the Einstein field equations* by means of which large classes of solutions can be constructed.

1.3 The Cauchy problem in general relativity

As in the case of many other physical theories, general relativity admits the formulation of an *initial value problem* (*Cauchy problem*). This aspect of the theory is obscured by both the *tensorial character of the Einstein field equations* and the *absence of a background geometry in the theory*; it is a priori not clear that the field equations give rise to a system of partial differential equations (PDEs) of a recognisable type.

Classical physical theories are expected to satisfy a ***causality principle****: the future of an event in spacetime cannot influence its past, and, moreover, signals must propagate at finite speed*. Among the three main types of PDEs (elliptic, hyperbolic and parabolic), *hyperbolic differential equations* are the only ones compatible with the causality principle. This observation suggests it should be possible to extract from the Einstein field equations a system of evolution equations with *hyperbolic properties*.

1.3.1 Hyperbolic reductions

The seminal work of Fourès-Bruhat (1952) has shown that the hyperbolic properties of the Einstein field equations can be made manifest by means of a suitable choice of coordinates. Following modern terminology, a choice of coordinates is a particular example of *gauge choice*. Indeed, by choosing the spacetime coordinates (x^μ) in such a way that they satisfy the wave equation associated with the metric $\tilde{\boldsymbol{g}}$, the Einstein field equations can be shown to imply a *system of quasilinear wave equations* for the components $(\tilde{g}_{\mu\nu})$ of the (a priori unknown) metric $\tilde{\boldsymbol{g}}$ with respect to the *wave coordinates*. For quasilinear wave differential equations there exists a developed theory which allows the formulation of a *well-posed Cauchy problem*. The use of wave coordinates is not the only way of bringing to the fore the hyperbolic aspects of the Einstein field equations. In this book, it will be shown that the Einstein field equations can be reformulated in such a way that after a suitable gauge choice they imply a so-called (first order) *symmetric hyperbolic evolution system* – a class of PDEs with properties similar to those of wave equations and for which a comparable theory is available. The procedure of extracting suitable hyperbolic *evolution equations* through a particular reformulation of the Einstein field equations and a suitable gauge choice is known as a ***hyperbolic reduction***; hyperbolic reductions are further discussed in Chapter 13. Besides its natural relevance in mathematical relativity, the construction of hyperbolic reductions for the Einstein field equations is of fundamental importance for numerical relativity; see, for example, Alcubierre (2008) and Baumgarte and Shapiro (2010).

In the same way that the Einstein field equations are geometric in nature, a proper formulation of the Cauchy problem in general relativity must also be done in a geometric way; see, for example, Choquet-Bruhat (2007). This idea is, in principle, in conflict with the discussion of hyperbolicity properties of the Einstein field equations, as the associated procedure of gauge fixing breaks the *spacetime covariance* of the field equations. As will be seen in the following, this tension can be resolved in a satisfactory manner.

1.3.2 Initial data and the constraint equations

The formulation of an initial value problem for the Einstein field equations requires the prescription of suitable initial data for the evolution equations on a three-dimensional manifold $\tilde{\mathcal{S}}$. This manifold will be later interpreted as a hypersurface of the spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$. An important feature of general relativity is that the initial data for the evolution equations implied by the Einstein field equations are constrained. The ***constraint equations of general relativity*** (*Einstein constraints*) can be formulated as a set of equations intrinsic to the initial hypersurface $\tilde{\mathcal{S}}$ for a pair of symmetric tensors $\tilde{\boldsymbol{h}}$ and $\tilde{\boldsymbol{K}}$ describing, respectively, the intrinsic geometry of the hypersurface (***intrinsic metric*** or ***first fundamental form***) and the way the initial hypersurface is curved within the spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ – the so-called ***extrinsic curvature*** or ***second fundamental form***. A priori, it is not clear what the *freely specifiable data* for these constraint equations consist of, or whether, given a particular choice of free data, the equations can be solved. The systematic analysis of the constraint equations has shown that under suitable assumptions, they can be recast as a set of *elliptic partial differential equations*; see, for example, Bartnik and Isenberg

(2004). For this type of equation a theory is available to discuss the existence and uniqueness of solutions.

The constraint equations play a fundamental role in the theory and ensure that the solution of the evolution equations is, in fact, a solution to the Einstein field equations; this type of analysis is often called the *propagation of the constraints*. The constraint equations of general relativity will be discussed in Chapter 11.

1.3.3 The well-posedness of the Cauchy problem in general relativity

The formulation of the Cauchy problem in general relativity ensures, at least locally, the existence of a solution to the Einstein field equations which is consistent with the prescribed initial data. More precisely, one has the following result first proven in Fourès-Bruhat (1952).

Theorem 1.1 (***local existence of solutions to the initial value problem***) *Given a solution* $(\tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ *to the Einstein constraint equations on a three-dimensional manifold* $\tilde{\mathcal{S}}$ *there exists a vacuum spacetime* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ *such that* $\tilde{\mathcal{S}}$ *is a spacelike hypersurface of* $\tilde{\mathcal{M}}$, $\tilde{\boldsymbol{h}}$ *is the intrinsic metric induced by* $\tilde{\boldsymbol{g}}$ *on* $\tilde{\mathcal{S}}$ *and* $\tilde{\boldsymbol{K}}$ *is the associated extrinsic curvature.*

The spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ obtained as a result of Theorem 1.1 is called a ***development of the initial data set*** $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$. Not every spacetime can be *globally* constructed from an initial value problem. Those which can be constructed in this way are said to be ***globally hyperbolic***. There are important examples of spacetimes which do not possess this property – most noticeably, the *anti-de Sitter spacetime*. A general result concerning globally hyperbolic spacetimes states that their topology is that of $\mathbb{R} \times \tilde{\mathcal{S}}$ with each slice $\tilde{\mathcal{S}}_t \equiv \{t\} \times \tilde{\mathcal{S}}$ being intersected only once by each timelike curve in the spacetime. The slices $\tilde{\mathcal{S}}_t$ are known as ***Cauchy surfaces***. The above points will be further discussed in Chapter 14.

The Cauchy problem for the Einstein field equations provides an appropriate setting for the discussion of dynamics. In particular, it allows one to investigate whether a given solution of the Einstein field equations is *stable*, that is, whether its essential features are retained if the initial data set is perturbed. Moreover, it also allows one to analyse whether a given property of a solution is *generic*, that is, whether the property holds for all solutions in an open set in the *space of initial data*.

1.3.4 Geometric uniqueness and the maximal globally hyperbolic development

An important observation concerning Theorem 1.1 is that it does not ensure the uniqueness of the development $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ of the initial data set $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$: a different hyperbolic reduction procedure will, in general, give rise to an alternative development $(\tilde{\mathcal{M}}', \tilde{\boldsymbol{g}}')$. From the point of view of the Cauchy problem of general relativity, the solution manifold is not known a priori. Instead, it is obtained as a part of the evolution process.

Given that an initial data set for the Einstein field equations gives rise to an infinite number of developments (one for each *reasonable* gauge choice), it is natural to ask whether it is possible to combine these various developments to obtain a *maximal development*. This question is answered in the positive by the following fundamental result; see Choquet-Bruhat and Geroch (1969).

Theorem 1.2 (***existence of a maximal development***) *Given an initial data set for the Einstein field equations* $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$*, there exists a unique maximal development* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$*, that is, a development such that if* $(\tilde{\mathcal{M}}', \tilde{\boldsymbol{g}}')$ *is another development, then* $\tilde{\mathcal{M}}' \subseteq \tilde{\mathcal{M}}$ *and on* $\tilde{\mathcal{M}}'$ *the metrics* $\tilde{\boldsymbol{g}}$ *and* $\tilde{\boldsymbol{g}}'$ *are isometric.*

The *maximal development* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ is also known as the *maximal globally hyperbolic development* of the data $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$. Theorem 1.2 clarifies the sense in which one can expect uniqueness from the Cauchy problem in general relativity; this idea is known as *geometric uniqueness*.

One can think of the maximal development of an initial data set as the largest spacetime that can be uniquely constructed out of an initial value problem. The boundary of this maximal development, if any at all, sets the limits of predictability of the data – accordingly, one has a close link with the notion of *classical determinism*. In certain spacetimes, it is possible to extend the maximal development of a hypersurface to obtain a *maximal extension*. Accordingly, in general, maximal developments and maximal extensions do not coincide. A further discussion of the Cauchy problem in general relativity is provided in Chapter 14.

1.3.5 Construction of maximal developments and global existence of solutions

Given some initial data set $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$, it is natural to ask, How can one construct its maximal development $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$? In general, this is a very difficult task, as it requires controlling the evolution dictated by the Einstein field equations under very general circumstances – something for which the required mathematical technology is not yet available. There are, nevertheless, some conjectures concerning the global behaviour of maximal developments. The origin of these conjectures goes back to Penrose (1969) – see Penrose (2002) for a reprint – and are usually known by the name *cosmic censorship*. In particular, the so-called *strong cosmic censorship* states that the maximal development of generic initial data for the Einstein field equations cannot be extended as a Lorentzian manifold.

Given an exact solution to the Einstein equations, if one knows its maximal extension, one can determine the maximal development $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ of one of its (Cauchy) hypersurfaces, say, $\tilde{\mathcal{S}}$. In what follows, let $(\tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ denote the initial data implied on $\tilde{\mathcal{S}}$ by the spacetime metric $\tilde{\boldsymbol{g}}$. The explicit knowledge of the maximal development allows one to provide a physical interpretation of the solution and to analyse its global structure in some detail. One can now ask whether certain aspects of $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ – say, its basic global structure – are shared by a wider class of solutions to the Einstein field equations. A strategy to address this question within the framework of the Cauchy problem in general relativity is to consider initial data sets $(\tilde{\mathcal{S}}, \bar{\boldsymbol{h}}, \bar{\boldsymbol{K}})$ which are, in some sense, close to the initial data for the exact solution. One can then try to show that the associated maximal globally hyperbolic development $(\bar{\mathcal{M}}, \bar{\boldsymbol{g}})$ has the desired global properties. If this is the case, one has obtained a statement about the *stability* of the solution and the *genericity* of the property one is interested in. The standard convention, to be used in this book, is to call $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ and $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$, respectively, the ***background spacetime*** and the ***background initial data set*** and $(\bar{\mathcal{M}}, \bar{\boldsymbol{g}})$ and $(\tilde{\mathcal{S}}, \bar{\boldsymbol{h}}, \bar{\boldsymbol{K}})$ the ***perturbed spacetime*** and ***perturbed initial data set***, respectively. In practice, the notion of closeness between initial data sets is dictated by the requirements of the PDE theory used to prove the existence of solutions to the evolution equations. In the previous discussion it has been assumed that the 3-manifolds on which the background and perturbed initial data are prescribed are the same. The stability analysis allows one to conclude that the spacetime manifolds $\tilde{\mathcal{M}}$ and $\bar{\mathcal{M}}$ are the same – they are, however, endowed with different metrics.

In analysing the stability of the background solution $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ one needs to show that the solutions to the evolution equations with perturbed initial data exist as long as the background solution. The expectation is that the assumption of having initial data close to data for an exact solution whose global structure is well understood will ease this task. In the following sections a strategy to exploit this assumption will be discussed.

1.4 Conformal geometry and general relativity

Special relativity provides a framework for the discussion of the notion of ***causality*** – that is, the relation between cause and effect – which is consistent with the *principle of relativity*. The *causal structure* of special relativity is determined by the light cones associated with the Minkowski metric $\tilde{\boldsymbol{\eta}}$. It allows the determination of whether a signal travelling not faster than the speed of light can be sent between two events – if this is the case, then the two events are said to be ***causally related***. More generally, one can talk of *Lorentzian causality*: any Lorentzian metric $\tilde{\boldsymbol{g}}$ gives rise to a causal structure determined by the light cones associated to $\tilde{\boldsymbol{g}}$. Thus, general relativity provides a natural generalisation of the notions of causality of special relativity – one in which the light cones vary from event to event in spacetime. Crucially, however, in general relativity the causal structure is a basic unknown of the theory.

The theory of hyperbolic differential equations provides notions of causality which, in principle, are independent from the notions of Lorentzian causality. It is, nevertheless, a remarkable feature of general relativity that locally, the propagation of fields dictated by the Einstein field equations is governed by the structure of the light cones of the solutions – the so-called *characteristic* *surfaces* of the evolution equations. Thus, the notions of Lorentzian and PDE causality coincide. This aspect of the Einstein field equations is further discussed in Chapter 14.

1.4.1 Conformal transformations and conformal geometry

Locally, a light cone can be described (away from its vertex) in terms of a condition of the form $\phi(x^\mu) = constant$ where $\phi : \tilde{\mathcal{M}} \to \mathbb{R}$ is such that

$$\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 0. \tag{1.4}$$

The structure of the light cones of a spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ is preserved by ***conformal rescalings***, that is, transformations of the spacetime metric of the form

$$\tilde{\boldsymbol{g}} \mapsto \boldsymbol{g} \equiv \Xi^2\tilde{\boldsymbol{g}}, \qquad \Xi > 0 \tag{1.5}$$

where Ξ is a smooth function on $\tilde{\mathcal{M}}$ – the so-called ***conformal factor***. Throughout this book, the metrics $\tilde{\boldsymbol{g}}$ and $\boldsymbol{g}$ will be called the ***physical metric*** and the ***unphysical metric***, respectively. The rescaling (1.5) gives rise to a ***conformal transformation*** of $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ to $(\tilde{\mathcal{M}}, \boldsymbol{g})$. Precise definitions and further discussion of these notions are provided in Chapter 5. In elementary geometry, conformal transformations are usually described as transformations preserving the angle between vectors. In Lorentzian geometry, they preserve the light cones; from (1.4) it follows that $g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 0$, so that the condition $\phi(x^\mu) = constant$ also describes the light cones of the metric $\boldsymbol{g}$.

One key aspect of conformal rescalings is that they allow one to introduce *conformal extensions* of the spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$; see Figure 1.1. In a Riemannian setting, the most basic example of conformal extensions of manifolds is the so-called *conformal completion* of the Euclidean plane $\mathbb{R}^2$ into the 2-sphere $\mathbb{S}^2$ by

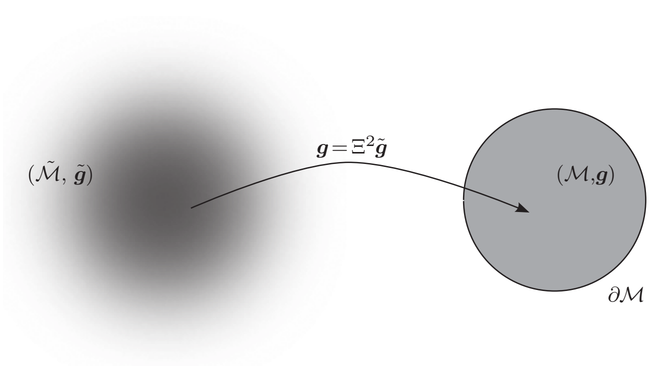

Figure 1.1 Schematic representation of the conformal extension of a manifold. The *physical* manifold $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ has infinite extension, while the *unphysical* (extended) manifold $(\mathcal{M}, \boldsymbol{g})$ is compact with boundary $\partial\mathcal{M}$. The boundary $\partial\mathcal{M}$ corresponds to the points for which $\Xi = 0$. Further details can be found in Chapter 5. Adapted from Penrose (1964).

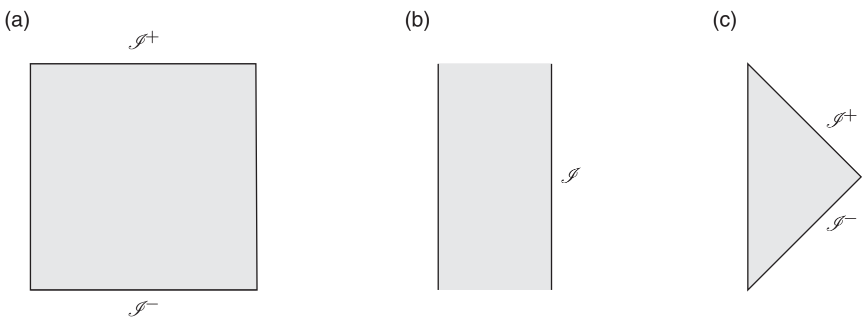

Figure 1.2 Penrose diagrams of the three spacetimes of constant curvature: (a) the de Sitter spacetime; (b) the anti-de Sitter spacetime; (c) the Minkowski spacetime. Details of these constructions can be found in Chapter 6.

means of stereographic coordinates. By suitably choosing the conformal factor Ξ, the metric $\boldsymbol{g}$ given by the rescaling (1.5) may be well defined even at the points where $\Xi = 0$. If this is the case, it can be verified that the set of points $\partial\mathcal{M}$ for which $\Xi = 0$ corresponds to *ideal points at infinity* for the spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ and is called the ***conformal boundary***. The pair $(\mathcal{M}, \boldsymbol{g})$ where $\mathcal{M}$ is the extended manifold obtained from attaching to $\tilde{\mathcal{M}}$ its conformal boundary is usually known as the ***unphysical spacetime***. Of particular interest are the portions of the conformal boundary which are hypersurfaces of the manifold $\mathcal{M}$ – these sets are characterised by the additional requirement of $\mathbf{d}\Xi \neq 0$, so that they have a well-defined normal. This part of the conformal boundary is denoted by $\mathscr{I}$.

Explicit calculations show that the three spacetimes of *constant curvature* – the Minkowski, de Sitter and anti-de Sitter spacetimes – can be conformally extended. The details of these constructions are described in Chapter 6. These conformal extensions are conveniently represented in terms of *Penrose diagrams*; see Figure 1.2. A discussion of the construction of Penrose diagrams can also be found in Chapter 6. The insights provided by the conformal extensions of these solutions are, in great measure, the fundamental justification for the use of conformal methods in general relativity.

1.4.2 Conformal geometry

The study of properties which are invariant under conformal transformations of a manifold is known as ***conformal geometry***. Associated to the metric $\boldsymbol{g}$ of the unphysical spacetime $(\mathcal{M}, \boldsymbol{g})$ one has its covariant derivative (connection) ∇_a and its curvature tensors, say, $R^a{}_{bcd}$, R_{ab}, R. These objects can be related to the corresponding objects associated to the physical metric $\tilde{\boldsymbol{g}}$ ($\tilde{\nabla}_a$, $\tilde{R}^a{}_{bcd}$, $\tilde{R}_{ab}$ and $\tilde{R}$) and the conformal factor Ξ and its derivatives. Their transformation laws show, in particular, that the Riemann tensor, the Ricci tensor and the Ricci scalar are not conformal invariants. There is, however, another part of the curvature which