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Basic concepts

Magnetic phenomena have been known since antiquity when a natural ore later called lodestone was discovered to attract bits of iron. The scientific study of magnetism dates from around 1600, when William Gilbert summarized experiments on the subject in his treatise *De Magnete*.[1] However, interest in the subject greatly increased after 1820, when Hans Christian Øersted reported that electrical currents could deflect magnetic needles, thereby establishing a connection between the subjects of electricity and magnetism.[2] Almost immediately, André-Marie Ampère, Jean-Baptiste Biot, and Félix Savart performed a series of seminal experiments that determined the forces acting between current loops. Experimental work and theoretical developments continued throughout the first half of the nineteenth century. A long program of experimental investigations by Michael Faraday lead him to the conception that the force between current loops occurred through the action of an intermediary field that existed in the space around the loops. Faraday's field concept was developed mathematically by William Thomson (later Lord Kelvin). This work culminated in a synthesis of knowledge about electrical and magnetic phenomena by James Clerk Maxwell in his famous treatise of 1873. Many clarifications of Maxwell's ideas and studies of their implications were carried out over the next twenty years by a small group of followers. Of particular note was the work of Oliver Heaviside who introduced the use of vector analysis and reworked the set of equations in Maxwell's treatise to the four equations we use today.[3] The resulting Maxwell equations are now accepted as the theoretical description underlying electromagnetic phenomena.

Magnetostatics is the study of the fields, forces, and energy associated with steady currents and magnetic materials. In this chapter, we will review some basic concepts underlying magnetic effects due to conductor currents in free space.

1.1 Current

Experiments have shown that there exist two kinds of electrical charge q, which are denoted as positive and negative. A current I exists when there is a net temporal flow of charge across some arbitrary plane in space.

$$I = \frac{dq}{dt}. \tag{1.1}$$

If the current is flowing through a conductor with length L and cross sectional area A, we can write the current as

$$I = \frac{\rho L A}{L/v} = \rho v A,$$

where ρ is the charge density and v is the velocity of the charges. The current density J along some direction n is a vector given by

$$\vec{J} = \frac{I}{A}\hat{n} = \rho\vec{v}, \tag{1.2}$$

where $\hat{n}$ is the unit vector perpendicular to A.

If we consider a volume of space V enclosed by a surface S, the conservation of charge requires that any change in the charge density inside V must be compensated by a flow of current through the surface or

$$-\int \frac{\partial \rho}{\partial t} dV = \int \vec{J}\cdot\hat{n}\, dS.$$

Using the Gauss divergence theorem,[1] the right-hand side can be written as

$$\int \vec{J}\cdot\hat{n}\, dS = \int \nabla\cdot\vec{J}\, dV.$$

Then, since V is arbitrary, we can remove the integrands from the volume integrals on both sides of the equation and obtain the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\vec{J} = 0. \tag{1.3}$$

In magnetostatics, we have by definition $\partial\rho/\partial t = 0$, which leads to the relation

$$\nabla\cdot\vec{J} = 0. \tag{1.4}$$

[1] Readers unfamiliar with vector analysis should review Appendix B.

Often we are interested in the current flow in a "central" region far from the ends of a magnet. If the current and the geometry are uniform along z in this region, we can simplify the analysis by examining problems in two dimensions. If we consider a conductor whose thickness is small compared with the distance to the observation point, we can approximate the conductor as a *current sheet*.

In addition, we frequently consider line currents or "filaments," where we ignore the transverse dimensions of the conductor altogether and use the equivalent current

$$I = \int \overrightarrow{J} \cdot \hat{n}\, dS.$$

1.2 Magnetic forces

Experiments have shown that test currents and charges in the vicinity of a current-carrying conductor experience a force. We assume that this force takes place through the actions of an intermediary magnetic field. The mathematical description of a field is a continuous function that is defined for all points in space and for all times. However, the magnetic field also has physical properties associated with it, such as stored energy. The force experiments can be explained by assuming that a current produces a vector field B, and then this field produces a force on other currents and charges. The vector field B is called the *magnetic flux density*[2] or magnetic field for short. The *magnetic flux* through some surface S is defined as

$$\Phi_B = \int \overrightarrow{B} \cdot \overrightarrow{dS}\,. \tag{1.5}$$

The direction of the magnetic field is often represented using Faraday's concept of *lines of induction.*[3][4] The lines of induction are defined to be tangent to the magnetic field at every point in space. It follows that corresponding components of the lines of induction and the magnetic field are always proportional to each other. If ds is a small displacement along the line of induction, we have

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z} = \frac{ds}{B}.$$

In two Cartesian dimensions, the lines can be plotted, for example, by integrating the equations

[2] The vector B is also known as the magnetic induction.
[3] Historically, these curves have been referred to as lines of force.

$$dx = \frac{B_x(x,y)}{B(x,y)}ds$$
$$dy = \frac{B_y(x,y)}{B(x,y)}ds.$$

The magnitude of the magnetic field can be represented by the density of lines in a given region. The lines of induction do not have to form closed loops.[5, 6] In particular, the lines become undefined at locations where $B = 0$.

Now consider two circuits carrying currents I_a and I_b. The force exerted by circuit a on circuit b is found experimentally to be

$$\vec{F}_{ab} = \frac{\mu_0}{4\pi} I_a\, I_b \oint\!\!\oint \overrightarrow{dl_b} \times \frac{\overrightarrow{dl_a} \times \vec{R}}{R^3}, \tag{1.6}$$

where the constant $\mu_0 = 4\pi\ 10^{-7}$ is known as the *permeability of free space*,[4] dl is a displacement along the circuit in the direction of the current, and R is the distance vector from dl_a to dl_b. Note that the force is proportional to the product of the currents times a geometric factor that depends on the shape and orientations of the two circuits. It is possible to rewrite this equation in a form that manifestly obeys Newton's Third Law of motion. Using the vector triple product identity from Equation B.1 in Appendix B, we have

$$\overrightarrow{dl_b} \times (\overrightarrow{dl_a} \times \vec{R}) = \overrightarrow{dl_a}\,(\overrightarrow{dl_b} \cdot \vec{R}) - \vec{R}\,(\overrightarrow{dl_a} \cdot \overrightarrow{dl_b}).$$

The double integral of the first term on the right-hand side is then

$$\oint\!\!\oint \frac{\vec{dl}_a(\vec{dl}_b \cdot \vec{R})}{R^3} = \oint \vec{dl}_a \oint \frac{(\vec{dl}_b \cdot \vec{R})}{R^3} = \oint \vec{dl}_a \oint \frac{dR}{R^2}.$$

The last integral vanishes because the scalar integrand is taken over a closed path. Thus we can express the force as

$$\vec{F}_{ab} = -\frac{\mu_0}{4\pi} I_a I_b \oint\!\!\oint \frac{\vec{R}\,(\vec{dl}_a \cdot \vec{dl}_b)}{R^3}. \tag{1.7}$$

In this form, we see that Newton's law $F_{ab} = -F_{ba}$ is obeyed since R changes direction for the two cases.

Returning to Equation 1.6, we rewrite the force on circuit b in a form that explicitly depends on the current in circuit b and on an integration of the elemental

[4] We will use SI units exclusively in this book. For more details, see Appendix A.

interactions taking place around that circuit. We collect the other factors in Equation 1.6 into a new vector B_a, which we define as the magnetic field due to circuit a. Then the force on the circuit can be written as

$$\vec{F}_{ab} = I_b \oint \vec{dl}_b \times \vec{B}_a \,. \tag{1.8}$$

The force acts at right angles to the direction of B_a. Dropping the subscripts, we see that the force on a charge q moving with velocity v can be written as

$$\vec{F} = \int \frac{dq}{dt} \vec{dl} \times \vec{B} = q\vec{v} \times \vec{B} \,. \tag{1.9}$$

Note that the force only acts on moving charges.

Now consider a rectangular current loop with length L and width w in a constant magnetic field B, as shown in Figure 1.1. The forces on each pair of opposite sides cancel, so there is no net force on the loop. However, there are moment arms between sides 1 and 3 and the axis of the loop. This creates a torque given by

$$\vec{\tau} = \vec{r} \times \vec{F} \,.$$

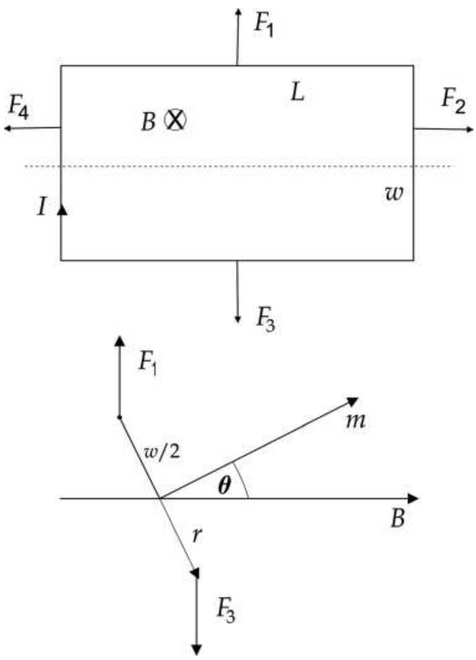

Figure 1.1 Rectangular current loop in an external field.

For the example here,

$$\tau = 2\frac{w}{2}NILB\sin\theta,$$

where N is the number of turns in the loop. We define the *magnetic moment* $\boldsymbol{m}$ of a planar loop to lie along the normal n to the loop, so that

$$\vec{m} = NIA\,\hat{n}, \tag{1.10}$$

where A is the area of the loop. Then the torque acting on the loop can be expressed as

$$\vec{\tau} = \vec{m} \times \vec{B}. \tag{1.11}$$

1.3 The Biot-Savart law

Comparing Equations 1.6 and 1.8, we see that the force experiments require that the magnetic field can be expressed in the form

$$\vec{B} = \frac{\mu_0}{4\pi} I \oint \frac{\vec{dl} \times \vec{R}}{R^3}, \tag{1.12}$$

where we have dropped the subscripts referring to circuit a. The vector R points from the current element source to the observation (or field) point where the magnetic field is determined. This relation, known as the *Biot-Savart Law*, is an important tool for finding analytic and numerical solutions for the magnetic field produced by known current distributions. For a surface distribution of current, the total current in the Biot-Savart law can be generalized to give

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{K} \times \vec{R}}{R^3} dS, \tag{1.13}$$

where K is the surface current density. Likewise, for a volume distribution of current, we have

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times \vec{R}}{R^3} dV. \tag{1.14}$$

It is important to keep in mind that the Biot-Savart law and many of the other mathematical laws that we will subsequently develop ultimately depend on the validity of the experimental results on magnetic forces.

We consider next several elementary applications of the Biot-Savart law that we will need to refer to later in this book.

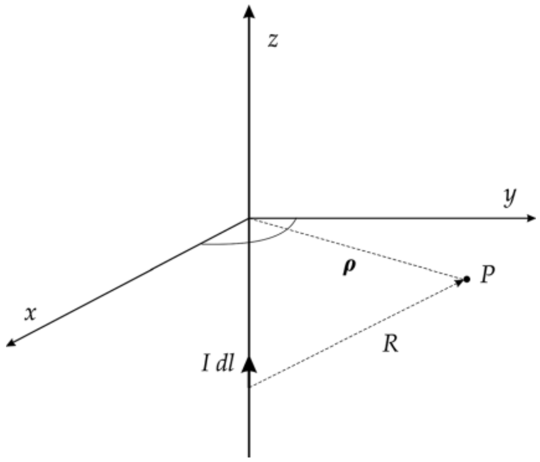

Figure 1.2 Current in a long straight wire.

Example 1.1: field from an infinitely long straight wire
Consider an infinitely long straight wire lying along the z axis, as shown in Figure 1.2. Because of the symmetry, we use cylindrical coordinates. Since the wire is infinitely long, we can chose an observation point P in the plane with $z = 0$ without loss of generality. Since

$$\begin{aligned}\overrightarrow{dl} &= dz\,\hat{z} \\ \overrightarrow{R} &= \rho\,\hat{\rho} + z\,\hat{z} \\ R &= \sqrt{\rho^2 + z^2},\end{aligned}$$

the field at point P due to the current in the wire is

$$\overrightarrow{B} = \frac{\mu_0}{4\pi}\rho\,\hat{\phi}\,2\,\mathbb{I},$$

where[5]

$$\mathbb{I} = \int_0^{\infty} \frac{dz}{\{\rho^2 + z^2\}^{3/2}} = \frac{1}{\rho^2}.$$

Thus the magnetic field due to the current in the wire is

$$\overrightarrow{B} = \frac{\mu_0 I}{2\pi\,\rho}\hat{\phi}. \tag{1.15}$$

The field is directed azimuthally around the wire and falls off with distance like $1/\rho$.

[5] GR 2.271.5.

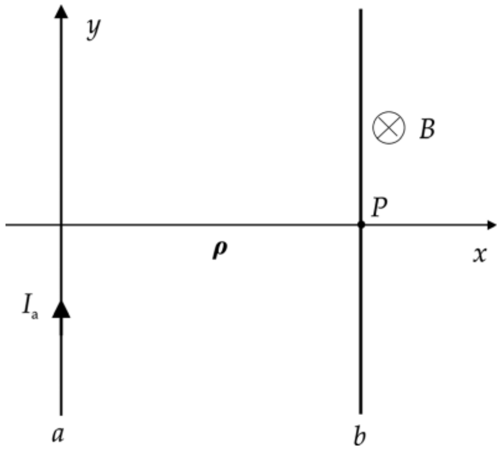

Figure 1.3 Force between two parallel wires.

Example 1.2: force between two parallel wires

Consider two infinitely long parallel wires, as shown in Figure 1.3. From Equation 1.8, the incremental force between the two wires is

$$\overrightarrow{dF}_b = I_b\, \overrightarrow{dl}_b \times \overrightarrow{B}_a$$

and from the previous example, the field at P due to the current in wire a is

$$\overrightarrow{B}_a = -\frac{\mu_0 I_a}{2\pi\rho}\hat{z}.$$

If the current direction in wire b can be either parallel or antiparallel to the current in wire a, we find that the force per unit length of the wire is

$$\frac{\overrightarrow{dF}_b}{dy} = \pm\frac{\mu_0}{2\pi\rho} I_a I_b\, \hat{x}. \tag{1.16}$$

The force between the wires is attractive when the currents are in the same direction and repulsive when they are antiparallel.

Example 1.3: field above an infinite current sheet

Consider an infinite current sheet with current flowing uniformly in the y direction. We calculate the magnetic field at point P, shown in Figure 1.4. We have

$$\begin{aligned}\overrightarrow{K} &= K_y\, \hat{y}\\ \overrightarrow{R} &= x\,\hat{x} + y\,\hat{y} + z_o\,\hat{z}.\end{aligned}$$

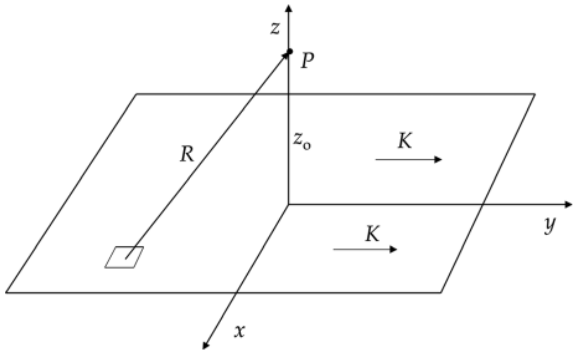

Figure 1.4 Field above an infinite current sheet.

The field is given by

$$\begin{aligned}\vec{B} &= \frac{\mu_0 K_y}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{z_o\,\hat{x}-x\,\hat{z}}{\{x^2+y^2+z_o^2\}^{3/2}}\,dx\,dy\\ &= \frac{\mu_0 K_y}{4\pi}(z_o\,\hat{x}\mathbb{I}_1-\hat{z}\mathbb{I}_2),\end{aligned}$$

where

$$\mathbb{I}_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\{x^2+y^2+z_o^2\}^{3/2}}\,dx\,dy = \frac{2\pi}{z_o}$$

and

$$\mathbb{I}_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x}{\{x^2+y^2+z_o^2\}^{3/2}}\,dx\,dy = 0.$$

The integral $\mathbb{I}_2$ vanishes because the integrand is an odd function and the integration extends over an even interval. The magnetic field above the sheet is

$$\vec{B} = \frac{\mu_0}{2}K_y\,\hat{x}.$$

The direction of the field is parallel to the sheet and perpendicular to the current density. The magnitude of the field is constant and independent of the distance from the sheet. In the general case, the field above the sheet can be written as

$$\vec{B} = \frac{\mu_0}{2}\vec{K}\times\hat{n}, \tag{1.17}$$

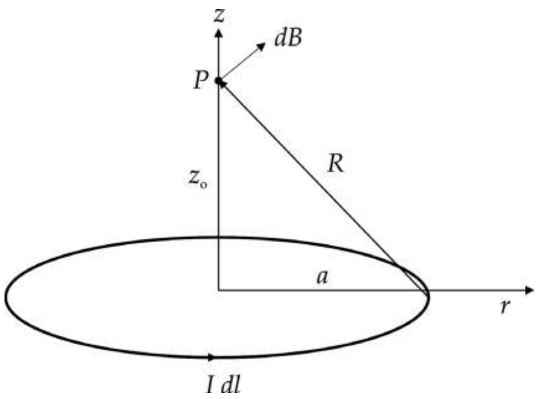

Figure 1.5 Field along the axis of a current loop.

where n is the normal to the sheet pointing to the side where B is computed. Note that the direction of B follows the right-hand rule with respect to the current filaments in the sheet.

Example 1.4: on-axis field due to a circular current loop
We look for the field at a point P that is along the axis of the loop and a distance z_o above the plane of the current loop, as shown in Figure 1.5. In cylindrical coordinates, we have

$$\begin{aligned}\vec{dl} &= a\,d\phi\,\hat{\phi}\\ \vec{R} &= -a\,\hat{r} + z_o\,\hat{z}.\end{aligned}$$

The contributions of the current elements to the field at P lie in a cone surrounding P. By symmetry, the net field must be in the z direction and

$$(\vec{dl} \times \vec{R})_z = a^2\,d\phi.$$

Thus we have

$$\begin{aligned}B_z &= \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\frac{a^2}{\{a^2+z_o^2\}^{3/2}}\,d\phi\\ &= \frac{\mu_0 I\,a^2}{2\{a^2+z_o^2\}^{3/2}}.\end{aligned} \tag{1.18}$$

Note that B_z is proportional to the area of the current loop and falls off at large distances like z_o^{-3}. The field is largest at the center of the loop where the value is

$$B_{z0} = \frac{\mu_0 I}{2a}.$$