Introduction

Rectifiable sets, measures, currents and varifolds are basic objects of geometric measure theory. In particular, during the last four decades, they have spread out to many areas of analysis and geometry. One of the goals of this survey is to show how rectifiability unifies surprisingly many different topics. Starting from the beginning and basic theories, I shall briefly describe many of these appearances. The table of contents should give a pretty good idea of what will follow. Here I just mention some of the milestones.

The pre-beginning is the right generalization of length, area, and so on. This was provided by Carathéodory in 1914. Lebesgue with his measure had given a generalization of volume in the Euclidean n-space and Carathéodory continued from this to define for integers $0 < m < n$ an outer measure generalizing the m-dimensional area. This measure $\mathcal{H}^m$ is now called m-dimensional Hausdorff measure, since Hausdorff generalized it further in 1919 to non-integral values of m. Once equipped with this tool, Besicovitch began in the 1920s to study the properties of $\mathcal{H}^1$ measurable planar sets E with $\mathcal{H}^1(E) < \infty$. In three papers he was able to reveal an amazing amount of structure. Such a set splits into a rectifiable and purely unrectifiable part. The former has properties similar to those of smooth surfaces and the latter completely opposite properties. Federer generalized most of Besicovitch's theory in 1947 to m-dimensional sets in $\mathbb{R}^n$. After the founding work of Besicovitch and Federer, the ingenious ideas of Marstrand and Preiss have been the most influential for the basic theory of rectifiability.

In the 1950s, De Giorgi described the structure of sets of finite perimeter in terms of rectifiable sets. This gave rectifiability a permanent place in the calculus of variations. First for sets of codimension one and then via Federer and Fleming's theory of normal and integral currents in 1960 for all dimensions. These are generalized surfaces and another class of them of fundamental importance over the years, rectifiable varifolds, was introduced by Almgren in the 1960s and developed by Allard in the 1970s. In the 1990s, Simon proved some fundamental results on the rectifiability of the singularities of minimal currents and harmonic maps.

Rectifiability has played a big role in complex and harmonic analysis. In the 1950s, Vitushkin anticipated this for the geometric description of removable sets of bounded complex analytic functions, which was fully confirmed much later in 1998 by David. In the 1990s, continuing from Jones's analyst's travelling salesman theorem, David and Semmes established the theory of uniform rectifiability and its connections to singular integrals and other topics of harmonic analysis.

Rectifiability has found a prominent place also outside Euclidean spaces. After some important work in the 1990s by Ambrosio, by Preiss and Tiser and by Kirchheim, Ambrosio and Kirchheim developed the theory of rectifiable sets and currents in metric spaces in 2000. In 2001 Franchi, Serapioni and Serra Cassano introduced the right notions of rectifiability in Heisenberg groups, which has led to an extensive theory in general Carnot groups.

This survey covers many topics, but all of them briefly and only scratching the surface. I am trying to give the reader a flavour of each of them without detailed proofs, often with ideas of the proofs, and often just presenting the results. There probably are many topics and articles that I have not, but should have, mentioned. And certainly this text is biased – I have concentrated more on topics I know best, including my own work.

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1
Preliminaries

1.1 Notation

We denote by $\mathcal{L}^n$ the Lebesgue measure in the Euclidean n-space $\mathbb{R}^n$. In a metric space X, $d(A)$ stands for the diameter of A, $d(A, B)$ the minimal distance between the sets A and B, and $d(x, A)$ the distance from a point x to a set A. The closed ball with centre $x \in X$ and radius $r > 0$ is denoted by $B(x, r)$ and the open ball by $U(x, r)$. In $\mathbb{R}^n$ we sometimes denote $B^n(x, r)$. The unit sphere in $\mathbb{R}^n$ is S^{n-1}. The Grassmannian manifold of linear m-dimensional subspaces of $\mathbb{R}^n$ is $G(n, m)$. It is equipped with an orthogonally invariant Borel probability measure $\gamma_{n,m}$. For $V \in G(n, m)$, we denote by P_V the orthogonal projection onto V.

For $A \subset X$, we denote by $\mathcal{M}(A)$ the set of non-zero finite Borel measures μ on X with support $\operatorname{spt}\mu \subset A$. We shall denote by $f_\#\mu$ the push-forward of a measure μ under a map $f\colon f_\#\mu(A) = \mu(f^{-1}(A))$. The restriction of μ to a set A is defined by $\mu \llcorner A(B) = \mu(A \cap B)$. The notation $\ll$ stands for absolute continuity.

The characteristic function of a set A is χ_A. By the notation $M \lesssim N$, we mean that $M \leq CN$ for some constant C. The dependence of C should be clear from the context. The notation $M \sim N$ means that $M \lesssim N$ and $N \lesssim M$. By c and C, we mean positive constants with obvious dependence on the related parameters.

1.2 Hausdorff Measures

For $m \geq 0$, the m-dimensional Hausdorff measure $\mathcal{H}^m = \mathcal{H}^m_d$ in a metric space (X, d) is defined by

$$\mathcal{H}^m(A) = \lim_{\delta\to 0}\inf\left\{\sum_{i=1}^{\infty}\alpha(m)2^{-m}d(E_i)^m : A \subset \bigcup_{i=1}^{\infty} E_i, d(E_i) < \delta\right\}.$$

Then $\mathcal{H}^0$ is the counting measure. Usually m will be a positive integer and then $\alpha(m) = \mathcal{L}^m(B^m(0,1))$, from which it follows by the isodiametric inequality that $\mathcal{H}^m = \mathcal{L}^m$ in $\mathbb{R}^m$. The isodiametric inequality says that among the subsets of $\mathbb{R}^m$ with a given diameter, the ball has the largest volume; see, for example, [203, 2.10.33]. For non-integral values of m the choice of $\alpha(m)$ does not really matter. We denote by dim the Hausdorff dimension. The *spherical Hausdorff measure* $\mathcal{S}^m$ is defined in the same way but using only balls as covering sets.

The lower and upper m-densities of $A \subset X$ are defined by

$$\Theta^m_*(A, x) = \liminf_{r\to 0} \alpha(m)^{-1} r^{-m}\mathcal{H}^m(A \cap B(x, r)),$$

$$\Theta^{*m}(A, x) = \limsup_{r\to 0} \alpha(m)^{-1} r^{-m}\mathcal{H}^m(A \cap B(x, r)).$$

The density $\Theta^m(A, x)$ is defined as their common value if they are equal.

We have

Theorem 1.1 *If A is $\mathcal{H}^m$ measurable and $\mathcal{H}^m(A) < \infty$, then*

$$2^{-m} \le \Theta^{*m}(A, x) \le 1 \textit{ for } \mathcal{H}^m \textit{ almost all } x \in A,$$

$$\Theta^{*m}(A, x) = 0 \textit{ for } \mathcal{H}^m \textit{ almost all } x \in X \setminus A.$$

When $m \le 1$ the constant 2^{-m} is sharp; for $m > 1$ the best constant is not known.

We also have

Theorem 1.2 *If $A \subset X$ is $\mathcal{H}^m$ measurable and $\mathcal{H}^m(A) < \infty$, then*

$$\lim_{\delta\to 0}\sup\{d(B)^{-m}\mathcal{H}^m(A \cap B) : x \in B, d(B) < \delta\} = 1 \textit{ for } \mathcal{H}^m \textit{ almost all } x \in A.$$

For general measures, we have

Theorem 1.3 *Let $\mu \in \mathcal{M}(X), A \subset X$, and $0 < \lambda < \infty$.*

(1) *If $\Theta^{*m}(A, x) \le \lambda$ for $x \in A$, then $\mu(A) \le 2^m\lambda\mathcal{H}^m(A)$.*
(2) *If $\Theta^{*m}(A, x) \ge \lambda$ for $x \in A$, then $\mu(A) \ge \lambda\mathcal{H}^m(A)$.*

For the above results, see [203, 2.10.17–19], [190, Section 2.2] or [321, Chapter 6].

We say that a closed set E is AD-m-regular (AD for Ahlfors and David) if there is a positive number C such that

$$r^m/C \leq \mathcal{H}^m(E \cap B(x,r)) \leq Cr^m \text{ for } x \in E, 0 < r < d(E).$$

A measure μ is said to be AD-m-regular if

$$r^m/C \leq \mu(B(x,r)) \leq Cr^m \text{ for } x \in \operatorname{spt}\mu, 0 < r < d(\operatorname{spt}\mu),$$

which means that $\operatorname{spt}\mu$ is an AD-m-regular set.

1.3 Lipschitz Maps

Since Lipschitz maps are at the heart of rectifiability, we state here some basic well-known facts about them. We say that a map $f\colon X \to Y$ between metric spaces X and Y is *Lipschitz* if there is a positive number L such that

$$d(f(x), f(y)) \leq Ld(x,y) \text{ for } x, y \in X.$$

The smallest such L is the Lipschitz constant of f, which is denoted by $\operatorname{Lip}(f)$.

Euclidean valued Lipschitz maps $f\colon A \to \mathbb{R}^k, A \subset X$, can be extended: there is a Lipschitz map $g\colon X \to \mathbb{R}^k$ such that $g|A = f$, see [203, 2.10.43–44] or [321, Chapter 7].

Any Lipschitz map $g\colon \mathbb{R}^m \to \mathbb{R}^k$ is almost everywhere differentiable by Rademacher's theorem, see [203, 3.1.6] or [321, 7.3].

There is the Lusin type property: if $f\colon A \to \mathbb{R}^k, A \subset \mathbb{R}^m$ is Lipschitz, then for every $\varepsilon > 0$ there is a C^1 map $g\colon \mathbb{R}^m \to \mathbb{R}^k$ such that

$$\mathcal{L}^m(\{x \in A\colon g(x) \neq f(x)\}) < \varepsilon, \tag{1.1}$$

see [203, 3.1.16].

2
Rectifiable Curves

Let us first have a quick look at rectifiable curves, concentrating on some facts that are relevant for the more general rectifiable sets.

By a curve in $\mathbb{R}^n$ we mean a continuous image of a line segment, $C = f([a,b])$. The curve is rectifiable if you can rectify it, that is, take hold of the endpoints and pull it straight. This is the same as to say that the curve has finite length. A standard definition of the length is that it is the total variation of f:

$$V_f(a,b) = \sup\left\{\sum_{j=1}^{k} |f(x_j) - f(x_{j-1})| \colon a = x_0 < x_1 < \cdots < x_k = b\right\}.$$

However, this depends on f since f could travel through some parts of C several times. For us the length of C will be the one-dimensional Hausdorff measure $\mathcal{H}^1(C)$ of C. It agrees with $V_f(a,b)$ if f is injective, or more generally if the set of points of C which are covered more than once has zero $\mathcal{H}^1$ measure.

If $V_f(a,b) < \infty$, then f is a function of bounded variation. Such functions have many well-known nice properties, but for us it is important to know that we can do better: if f is the arc-length parametrization of C, then f is Lipschitz. This is essential, in particular, in the case of higher-dimensional rectifiable sets.

Now let $C = f([a,b])$ be a rectifiable curve in $\mathbb{R}^n$ with a Lipschitz parametrization f. Here are some of its basic easily verifiable properties:

$$\text{Area formula}: \int_C N(f,y)\, d\mathcal{H}^1 y = \int_a^b \sqrt{f_1'(x)^2 + \cdots + f_n'(x)^2}\, dx, \tag{2.1}$$

where $N(f,y)$ is the number of points $x \in [a,b]$ with $f(x) = y$. So, in particular,

$$\mathcal{H}^1(C) = \int_a^b \sqrt{f_1'(x)^2 + \cdots + f_n'(x)^2}\, dx$$

if f is injective. The key for the proof is Rademacher's theorem (or Lebesgue's in the one-dimensional case); Lipschitz mappings are almost everywhere differentiable. Using also that $\sqrt{f_1'(x)^2 + \cdots + f_n'(x)^2}$ tells us how the derivative of f at x changes length, a rather elementary proof can be given. As expected, a higher-dimensional version also is valid and will be presented later. Hence the name area formula.

With the help of the area formula and again Rademacher's theorem, the following two properties are not too hard to verify:

$$\text{Tangents: } C \text{ has a tangent at } \mathcal{H}^1 \text{ almost every point } x \in C. \tag{2.2}$$

$$\text{Density: } \lim_{r\to 0} \frac{\mathcal{H}^1(C \cap B(x,r))}{2r} = 1 \text{ for } \mathcal{H}^1 \text{ almost all } x \in C. \tag{2.3}$$

In the plane, the length can be computed by counting the intersection points with lines:

$$\text{Crofton formula: } 2\mathcal{H}^1(C) = \int \operatorname{card}(C \cap L)\, dL. \tag{2.4}$$

Here the measure dL on lines can be obtained by parametrizing the lines as $\{te + a \colon t \in \mathbb{R}\}, e \in S^1, a \in e^\perp$, and integrating over e and a.

This formula is trivially checked when C is a line segment. The general case can be done using Rademacher's theorem and approximation by polygonal curves. Crofton proved this in 1868, which marked the beginning of integral geometry – unless you want to start at 1777 with Count Buffon and his needle.

In the beginning I said that a curve is rectifiable if it has finite length. But if we take Hausdorff measure as length, can we get from its finiteness the Lipschitz parametrization which was used above? Yes, we can, even in general metric spaces:

Theorem 2.1 *If X is a metric space and $C \subset X$ is a compact connected set with $\mathcal{H}^1(C) < \infty$, then there is a Lipschitz mapping $f \colon [0,1] \to X$ with $f([0,1]) = C$.*

For a rather easy proof in $\mathbb{R}^n$, see [147, Theorem I.1.8], and in the Hilbert space, [394, Lemma 3.7]. Here are some ideas. For each $\delta > 0$ choose a maximal δ separated subset A_δ of C. Connect with line segments all those pairs of points of A_δ that have distance at most 2δ and let C_δ be the union of these segments. Playing with some graphs shows that C_δ is a continuum which can be parametrized by a Lipschitz map $f_\delta \colon [0,1] \to \mathbb{R}^n$ with $\operatorname{Lip}(f_\delta) \lesssim \mathcal{H}^1(C)$.

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Finally use the Arzela–Ascoli theorem to get f as the limit of some sequence (f_{δ_j}).

Theorem 2.1 was proved by Eilenberg and Harrold in [187]. It is one of the reasons why the rectifiability theory often is much easier for one-dimensional sets. Another reason is compactness and lower semicontinuity:

Theorem 2.2 *If $C_k \subset B^n(0,1), k = 1,2,\ldots$ are continua, then there is a subsequence C_{k_j} converging in the Hausdorff distance to a continuum C with $\mathcal{H}^1(C) \leq \liminf_{j\to\infty} \mathcal{H}^1(C_{k_j})$.*

Again the proof is rather easy, see [190, Theorems 3.16 and 3.18].

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3

One-Dimensional Rectifiable Sets

Almost all of the theory of one-dimensional rectifiable sets in the plane was developed by Besicovitch in the three papers [62], [63] and [64]. Some generalizations are due to Morse and Randolph [352] and Moore [350]. For most of the proofs of the results in this chapter, see [158, 190]. Several ideas extend to higher dimensions, and in the next chapter we shall mainly discuss the new ideas needed. We consider here only subsets of the plane although essentially everything holds for one-dimensional rectifiable subsets of $\mathbb{R}^n$.

3.1 Definitions and Tangents

One-dimensional rectifiable sets generalize rectifiable curves and they have many properties that make this class flexible in a measure-theoretic sense and useful in many applications. In particular,

- Every rectifiable curve is a rectifiable set.
- Countable unions of rectifiable sets are rectifiable.
- Subsets of rectifiable sets are rectifiable.
- Sets of zero $\mathcal{H}^1$ measure are rectifiable.
- If for every $\varepsilon > 0$ there is a rectifiable set $F \subset E$ with $\mathcal{H}^1(E \setminus F) < \varepsilon$, then E is rectifiable.

These properties offer an immediate definition: E is 1-rectifiable if $\mathcal{H}^1$ almost all of it can be covered with rectifiable curves. We state it a bit differently:

Definition 3.1 A set $E \subset \mathbb{R}^2$ is 1-*rectifiable* if there are Lipschitz maps $f_i : \mathbb{R} \to \mathbb{R}^2, i = 1, 2, \ldots$ such that

$$\mathcal{H}^1\left(E \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R})\right) = 0.$$

Next we shall see that rectifiable sets have, in an appropriate sense, the properties of rectifiable curves which we presented in Chapter 2. This is actually rather easy, but more essentially only rectifiable sets have these properties. That is, each of the properties allows a converse statement.

We shall begin with tangents. Consider the following example. Let q_i, $i = 1, 2, \dots$ be all the points of the plane with rational coordinates and set $E = \bigcup_{i=1}^{\infty} S_i$ with $S_i = \{x \in \mathbb{R}^2 : |x - q_i| = 2^{-i}\}$. Then E is 1-rectifiable and $\mathcal{H}^1(E) < \infty$. But it is dense in $\mathbb{R}^2$. So how can it have any tangents? Obviously it cannot in the ordinary sense and we have to give a measure-theoretic definition of a tangent. For $a \in \mathbb{R}^2$, $s > 0$, $L \in G(2, 1)$, define the cone (angular sector)

$$X(a, L, s) = \left\{x \in \mathbb{R}^2 : d(x, L + a) < s|x - a|\right\}.$$

Definition 3.2 A line $L \in G(2, 1)$ is an *approximate tangent line* of a set $E \subset \mathbb{R}^2$ at a point $a \in \mathbb{R}^2$ if $\Theta^{*1}(E, a) > 0$ and for every $s > 0$,

$$\lim_{r \to 0} r^{-1} \mathcal{H}^1\Big(E \cap B(a, r) \setminus X(a, L, s)\Big) = 0.$$

It is convenient to define the approximate tangents as lines through the origin. Then the geometric approximate tangent at a is the translate by a.

Now we immediately have that the above union E of the circles S_i has an approximate tangent at almost all of its points: each S_i has an ordinary tangent at all of its points and at almost all points it is an approximate tangent of E by the density theorem, Theorem 1.1. Since rectifiable curves have tangents almost everywhere, essentially the same argument shows that every 1-rectifiable set with finite $\mathcal{H}^1$ measure has an approximate tangent at almost all of its points.

The converse is not very difficult either. I give the idea assuming that E has ordinary tangents almost everywhere. Then we can write E as a countable union of a set of measure zero and sets F for which there exist $s > 0$ and $L \in G(2, 1)$ such that $F \setminus X(a, L, s) = \emptyset$ for $a \in F$. The set F is such that the tangents in its points are close to a fixed line L. This implies that the restriction of the projection P_L to F is one-to-one with a Lipschitz inverse. Hence $F = (P_L|F)^{-1}(P_L(F))$ is rectifiable.

The case of approximate tangents causes technical difficulties, but the main idea is the same. So we have

Theorem 3.3 *If E is $\mathcal{H}^1$ measurable and $\mathcal{H}^1(E) < \infty$, then E is 1-rectifiable if and only if it has an approximate tangent line at almost all of its points.*

Definition 3.4 A set $E \subset \mathbb{R}^2$ is *purely* 1-*unrectifiable* if $\mathcal{H}^1(E \cap F) = 0$ for every 1-rectifiable set $F \subset \mathbb{R}^2$ (or, equivalently, $\mathcal{H}^1(E \cap C) = 0$ for every rectifiable curve C).