1

General Introduction

Turbulence is often defined as the chaotic state of a fluid. The example that immediately comes to mind is that of water: turbulence in water takes the form of eddies whose size, location, and orientation are constantly changing. Such a flow is characterized by a very disordered behavior difficult to predict and by the existence of multiple spatial and temporal scales. There are many experiments of everyday life where the presence of turbulence can be verified: the agitated motions of a river downstream of an obstacle, those of smoke escaping from a chimney, or the turbulence zones that one sometimes crosses in an airplane.

Experiencing turbulence at our scale seems easy since it is not necessary to use powerful microscopes or telescopes. A detailed analytical understanding of turbulence remains, however, limited because of the intrinsic difficulty of nonlinear physics. As a result, we often read that turbulence is one of the last great unresolved problems of classical physics. This long-held message, found, for example, in Feynman et al. (1964), no longer corresponds to the modern vision. Indeed, even if turbulence remains a very active research topic, we have to date many theoretical, numerical, experimental, and observational results that allow us to understand in detail a part of the physics of turbulence.

This book deals mainly with wave turbulence. However, wave turbulence is not totally disconnected from eddy turbulence, from which the main concepts have been borrowed (e.g. inertial range, cascade, two-point correlation function, spectral approach). Moreover, very often, wave turbulence and eddy turbulence can coexist as in rotating hydrodynamics. This is why a broad introduction to eddy turbulence is given (Part I) before moving on to wave turbulence (Part II), giving this book, for the first time, a unified view on turbulence. We will see that many results have been obtained since the first steps taken by Richardson (1922), a century ago. The many examples discussed in this book reveal that the classical presentation of turbulence, based on the Navier–Stokes equations (Frisch, 1995; Pope, 2000), is somewhat too simplistic because turbulence is found in various environments, in various forms. If we restrict ourselves to the standard example of incompressible hydrodynamics, the simple introduction of a uniform rotation for describing geophysical fluids drastically changes the physics of turbulence by adding anisotropy. In astrophysics, 99 percent of the visible matter of the Universe is in the form of plasma, which is generally very turbulent, but plasma turbulence mixes waves and eddies. The regime of wave turbulence, described in Part II, can emerge from a vibrating steel plate; here, we are far from the classical image of eddies in water. Finally, recent studies reveal that the cosmological inflation that followed the Big Bang could have its origin in strong gravitational wave turbulence.

The objective of Part I, which follows this first chapter, is to present the fundamentals of turbulence. We will start with eddy turbulence, where the first concepts and laws have emerged. We will limit ourselves to the most important physical laws. The theoretical framework will be that of a statistically homogeneous turbulence for which a universal behavior is expected. The problems of inhomogeneity inherent to laboratory experiments will therefore not be dealt with. Through the examples discussed, we will gradually reveal the state of knowledge in turbulence. To help us in this task, we begin with a brief historical presentation.

1.1 Brief History

1.1.1 First Cognitive Advances

Leonardo da Vinci was probably the first to introduce the word *turbulence* (*turbulenza*) at the beginning of the sixteenth century to describe the tumultuous movements of water. However, the word was not commonly used by scientists until much later.[1]

The first notable scientific breakthrough in the field of turbulence can be attributed to Reynolds (1883): he showed experimentally that the transition between the laminar and turbulent regimes was linked to a dimensionless number – the Reynolds number.[2] The experiment, which can be easily reproduced in a laboratory, consists of introducing a colored stream of the same liquid as circulating in a straight transparent tube (see Figure 1.1). It can be shown that the transition to turbulence occurs when the Reynolds number becomes greater than a critical value. An important step in this discovery is the observation that the tendency to form eddies increases with the temperature of the water, and Reynolds knew that in this case the viscosity decreases. He also showed the important role played by the development of instabilities in this transition to turbulence.

World War I was a time of further important advances. The war efforts in Germany and, in particular, under the influence of Prandt in Göttingen, directed

1 For example, the book of Boussinesq (1897) still bears the evocative title: "Theory of the Swirling and Tumultuous Flow of Liquids in Straight Beds with a Large Section."

2 The Reynolds number measures the ratio between the inertial force and the viscous force. We will come back to this definition in Section 1.3.

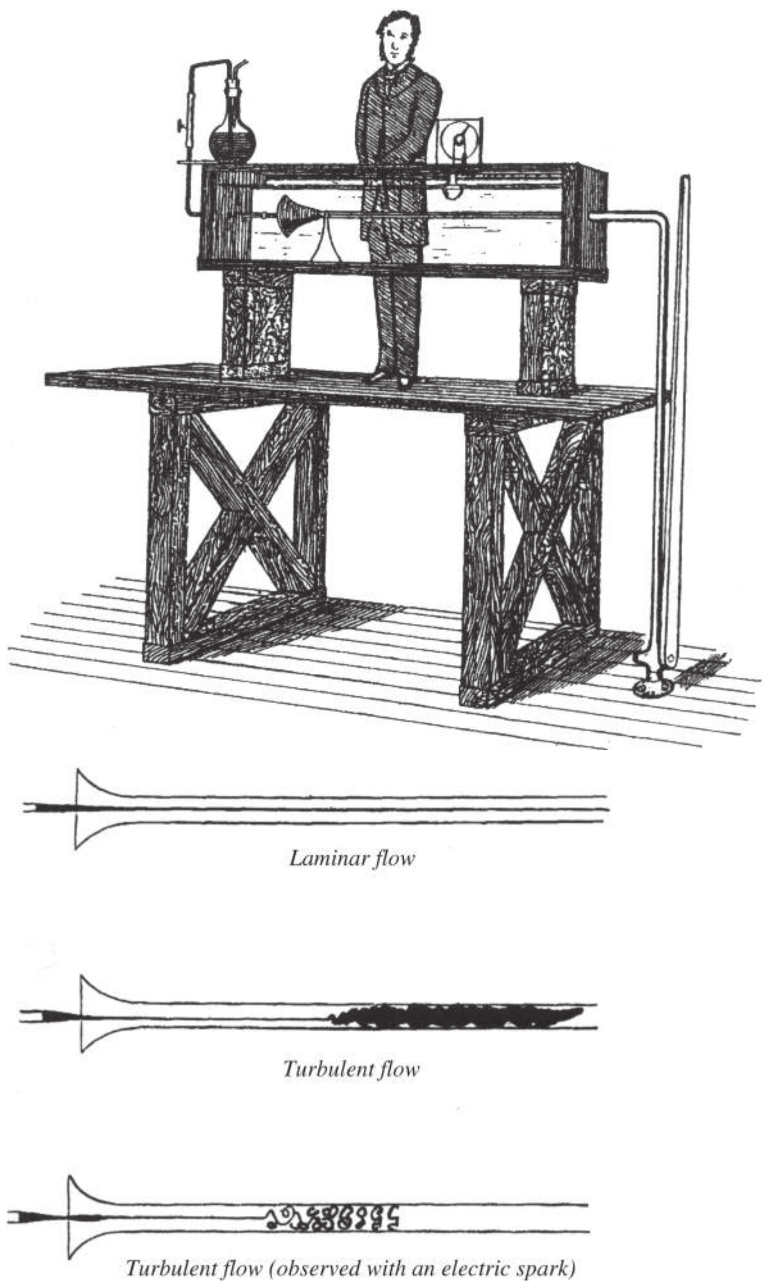

Figure 1.1 Historical experiment of Reynolds (1883) (top) and his observations (bottom). The original device is kept at the University of Manchester.

the research in the field of aerodynamics to the study of the fall of bombs in air or water. It is a question here of studying, for example, the drag of a sphere;

this work was then used for the design of airplanes. After the war, research in turbulence increased: for example, we can mention the results on the inhomogeneous effects due to walls in wind tunnel experiments (Burgers, 1925). But it is with Richardson (1922) that a second major breakthrough in turbulence arrives: in his book on weather predictions and numerical calculation.[3] Richardson introduced the fundamental concept of energy cascade. Inspired by the Irish writer J. Swift, Richardson wrote "Big whirls have little whirls that feed on their velocity. Little whirls have lesser whirls and so on to viscosity – in the molecular sense" (page 66). We find here the idea of a cascade of eddies from large to small spatial scales.

It is probably with this idea in mind that Richardson (1926) formulated the empirical 4/3 law[4] to describe the turbulent diffusion process. This law differs from the one proposed by Einstein in 1905 on the diffusion of small particles in a liquid (Brownian motion), which was in clear disagreement with turbulence experiments where a much higher diffusion was found.[5] The proposed new law is characterized by a nonconstant diffusion coefficient D_ℓ, which depends on the scale being considered, such that:

$$D_\ell \sim \ell^{4/3} \,. \tag{1.1}$$

This relationship reflects the fact that in a turbulent liquid the diffusivity increases with the mean separation between pairs of particles. This scaling law is fundamental because we find there the premises of the exact four-fifths law of Kolmogorov (1941a), with which it is in agreement dimensionally.

It was during this interwar period that the first works based on two-point correlations emerged (Taylor, 1935),[6] as well as works on the spectral analysis of fluctuations by Fourier transform, which have become the basis of modern research in turbulence (Motzfeld, 1938; Taylor, 1938). The correlation approach leads, in particular, to the Kármán–Howarth equation (von Kármán and Howarth,1938) for an incompressible, statistically homogeneous, and isotropic[7] hydrodynamic turbulence. This equation describes the fluid dynamics through correlators – two-point measurements in physical space. As we will see in Chapter 2, this result is central for the establishment of the exact four-fifths law of Kolmogorov (1941a), which is not a dynamic equation but a statistical solution of Navier–Stokes equations.

[3] "Numerical calculation" here means calculation carried out by hand with a method essentially based on finite differences.
[4] This empirical law should not be confused with the exact four-thirds law which deals with structure functions (see Chapter 2).
[5] It is known that a cloud of milk dilutes more rapidly in tea if stirred with a spoon.
[6] It is the British Francis Galton (1822–1911) who seems to have been the first to correctly introduce the concept of correlation for statistical studies in biology.
[7] This is the strong isotropy that is considered here, which we will return to in Section 1.4.

1.1.2 Kolmogorov's Law and Intermittency

In the 1930s and under the leadership of the mathematician Kolmogorov, the Soviet school became very active in turbulence. At that time, Kolmogorov was working on stochastic processes and random functions. It was therefore natural that he turned his attention to turbulence, where a pool of data was available. Based on some of the work described in the Section 1.1.1, Kolmogorov and his student Obukhov set out to develop a theory for the standard case of incompressible, statistically homogeneous, and isotropic hydrodynamic turbulence. Based, in particular, on the Kármán–Howarth equation, Kolmogorov (1941a,b) established the first exact statistical law of turbulence – known as the four-fifths law – which relates a third-order structure function involving the difference of the component in direction $\boldsymbol{\ell}$ of the velocity between two points separated by the vector $\boldsymbol{\ell}$, the distance ℓ, and the mean rate of dissipation of kinetic energy ε ($\langle\rangle$ means the ensemble average):[8]

$$-\frac{4}{5}\varepsilon\ell = \langle [u_\ell(\mathbf{x}+\boldsymbol{\ell}) - u_\ell(\mathbf{x})]^3 \rangle. \tag{1.2}$$

To establish this universal law, Kolmogorov assumes that fully developed turbulence becomes isotropic on a sufficiently small scale, regardless of the nature of the mean flow. He also assumes that ε becomes independent of viscosity within the limits of large Reynolds numbers (i.e. low viscosity); this is what is often referred to today as the zeroth law of turbulence. After several years of research, a first exact law was established for which it was possible to get rid of the nonlinear closure problem. The trick used to achieve this was to relate the cubic nonlinear term to the mean energy dissipation in the inertial range, that is, in a limited range of scales between the larger scales where inhomogeneous effects can be felt, and the smaller scales where viscosity efficiently damps the fluctuations. We will return at length to the law (1.2) in Chapter 2. Kolmogorov's law remained unnoticed for several years (outside the USSR). It was Batchelor (1946) who was the first to discover the existence of Kolmogorov's articles:[9] he immediately realized the importance of this work, which he shared with the scientific community at the Sixth International Congress of Applied Mathematics held in Paris in 1946 (Davidson et al., 2011).

For his part, independently of Kolmogorov but inspired by the ideas of Richardson (1922), Taylor (1938), and the work by Millionschikov (1939, 1941), who was another student of Kolmogorov, Obukhov (1941b) proposed a nonexact spectral theory of turbulence based on the relationship:

[8] Kolmogorov was probably the first to be interested in structure functions that are constructed from the differences and not from the products of a field (here the velocity field), as was the case with the Kármán–Howarth equation.

[9] The English version of the Russian papers had been received in the library of the Cambridge Philosophical Society.

$$\frac{\partial E}{\partial t} + D = T \,, \tag{1.3}$$

with E the energy spectrum, D the viscous dissipation, and T the energy transfer (in Fourier space). The artificial closure proposed is based on an average over small scales. He obtained as a solution the energy spectrum:[10]

$$E(k) \sim k^{-5/3} \,, \tag{1.4}$$

which is dimensionally compatible with Kolmogorov's exact law. In extending this study, Obukhov was then able to provide a theoretical justification for Richardson's (1926) empirical 4/3 law of diffusion. Later, Yaglom (1949) obtained a new exact law, applied this time to the passive scalar: this model describes how a scalar evolves, for example the temperature or the concentration of a product, in a turbulent fluid for which the velocity fluctuations are given.

For a short period of time Kolmogorov thought that the mean rate of energy dissipation was the key to establishing a more general exact law describing the statistics at any order in terms of a velocity structure function. This general law would have provided a complete statistical solution to the problem of hydrodynamic turbulence. But in 1944, Landau[11] pointed out the weakness of the demonstration (proposed by Kolmogorov during a seminar), which we will come back to in Chapter 2: it does not take into account the possible local fluctuations of ε, a property called intermittency. It took about 20 years for Kolmogorov (1962) and Oboukhov (1962) to propose, in response to Landau, a model (and not an exact law) of intermittency based on a log-normal statistics which incorporates the exact four-fifths law as a special case. Kolmogorov's answer was given (in French) at a conference held in Marseilles in 1961 to celebrate the opening of the Institut de Mécanique Statistique de la Turbulence. This conference became famous because it brought together for the first time all the major specialists (American, European, and Soviet) on the subject. It was also during this conference that the first energy spectrum in $k^{-5/3}$ measured at sea was announced (Grant et al., 1962).

Basically, the notion of intermittency is related to the concentration of dissipation in localized structures of vorticity. As mentioned by Kolmogorov, intermittency may slightly modify the $-5/3$ exponent of the energy spectrum, but its most important contribution is expected for statistical quantities of higher orders (the exact law is of course not affected). This new formulation is at the origin of work, in particular, on the concept of fractal dimension as a model of intermittency (Mandelbrot, 1974; Frisch et al., 1978) – see Chapter 2. It is interesting to note that we already find the concept of fractional dimension in Richardson's (1922) book, where the study of geographical boundaries is discussed.

10 In general, this solution is called the Kolmogorov spectrum, but it would be more accurate to call it the Kolmogorov–Obukhov spectrum. This spectrum was also obtained independently by other researchers, such as Onsager (1945) and Heisenberg (1948).

11 Landau's remark (Landau and Lifshitz, 1987) can be found in the original 1944 book (Davidson et al., 2011).

1.1.3 Spectral Theory and Closure

In this postwar period, the theoretical foundations of turbulence began to be established. The first book exclusively dedicated to this subject is that of Batchelor (1953), which still remains a standard reference on the subject: it deals with statistically homogeneous turbulence. From the 1950s, a major objective seemed to be within the reach of theorists: developing a theory for homogeneous and isotropic turbulence in order to rigorously obtain the energy spectrum. The work of Millionschikov (1941) (see also Chandrasekhar, 1955) based on the quasi-normal approximation (QN) had opened the way: this approximation – a closure – assumes that moments of order four and two are related as in the case of a normal (Gaussian) law without making this approximation for moments of order three (which would then be zero, making the problem trivial). Kraichnan (1957) was the first to point out that this closure was inconsistent because it violated some statistical inequalities (realizability conditions), and Ogura (1963) demonstrated numerically that this closure could lead to a negative energy spectrum for some wavenumbers.

In this quest, Kraichnan (1958, 1959) proposed a sophisticated theory which does not have the defects we have just mentioned: it is the direct interaction approximation (DIA), which is based on field theory methods, a domain in which Kraichnan was originally trained.[12] The fundamental idea of this approach is that a fluid perturbed over a wavenumber interval will have its perturbation spread over a large number of modes. Within the limit $L \to +\infty$, with L being the side of the cube in which the fluid is confined, this interval becomes infinite in size, which suggests that the mode coupling becomes infinitely weak. The response to the perturbation can then be treated in a systematic way. Under certain assumptions, two integro-differential equations are obtained for the correlation functions in two points of space and two of time, and the response function. The inferred prediction for the energy spectrum, in $k^{-3/2}$, is, however, not in dimensional agreement with Kolmogorov's theory, nor with the main spectral measurements. Improvements were then made (Lagrangian approach) to solve some problems (noninvariance by random Galilean transformation, Kolmogorov spectrum) (Kraichnan, 1966): this new theory can be seen as the most sophisticated closure model.[13] This work has led, in particular, to the development of the EDQNM (eddy-damped quasi-normal Markovian) closure model (Orszag, 1970), still widely used today, to which we will return in Chapter 3.

[12] Kraichnan became interested in turbulence in the early 1950s while he was Einstein's postdoctoral fellow. Together, they searched for nonlinear solutions to the unified field equations.

[13] In (strong) eddy turbulence, no exact spectral theory with an analytical closure has been found to date. This contrasts with the (weak) wave turbulence regime, for which an asymptotic closure is possible (see Chapter 4).

1.1.4 Inverse Cascade

Two-dimensional hydrodynamic (eddy) turbulence is the first example where an inverse cascade was suspected. The motivation for the study of such a system may seem on the face of it surprising, but several works showed that a two-dimensional approach could account for the atmospheric dynamics quite satisfactorily (Rossby and collaborators, 1939). We now know that the rotation, or stratification, of the Earth's atmosphere tends to confine its nonlinear dynamics to horizontal planes.[14] The first work on two-dimensional hydrodynamic turbulence dates back to the 1950s with, for example, Lee (1951), who demonstrated that a direct energy cascade would violate the conservation of enstrophy (proportional to vorticity squared), which is the second inviscid invariant (i.e. at zero viscosity) of the equations. Batchelor (1953) had also noted at the end of his book that the existence of this second invariant should contribute to the emergence, by aggregation, of larger and larger eddies. He concluded by asserting the very great difference between two- and three-dimensional turbulence. By using the two inviscid invariants, energy and enstrophy, Fjørtoft (1953) was able on his part to demonstrate, in particular with dimensional arguments, that the energy should cascade preferentially towards large scales.

It is in this context, clearly in favor of an inverse energy cascade, that Kraichnan became interested in two-dimensional turbulence. Using an analytical development of Navier–Stokes equations in Fourier space, the use of symmetries, and under certain hypotheses such as the scale invariance of triple moments, Kraichnan (1967) rigorously demonstrated the existence of a dual cascade – that is, in two different directions – of energy and enstrophy (see Chapter 3). This prediction is in agreement with previous analyses and the existence of a direct cascade of enstrophy and an inverse cascade of energy for which the proposed (nonexact) spectrum is in $k^{-5/3}$.

The existence in the same system of two different cascades was quite new in eddy turbulence. This prediction has since been accurately verified both experimentally and numerically (Leith, 1968; Pouquet et al., 1975; Paret and Tabeling, 1997; Chertkov et al., 2007). The second-best-known system where an inverse cascade exists is that of magnetohydrodynamics (MHD): using some arguments from Kraichnan (1967), Frisch et al. (1975) deduced in the three-dimensional case the possible existence of an inverse cascade of magnetic helicity, a quantity which plays a major role in the dynamo process in astrophysics (Galtier, 2016). To date, we know several examples of turbulent systems producing an inverse cascade (see, e.g., the review of Pouquet et al., 2019).

14 Chapter 6 is devoted to inertial wave turbulence (i.e. incompressible hydrodynamic turbulence under a uniform and rapid rotation), for which it can be rigorously demonstrated that the cascade is essentially reduced to the direction transverse to the axis of rotation. However, it can be shown in this case that the energy cascade is direct.

Kraichnan's (1967) discovery was made at a period when the theory of wave turbulence, the regime that is the main subject of this book, was beginning to produce important results. The brief history presented in Chapter 4 allows us to appreciate the evolution of ideas on this subject, which finds a large part of its foundations in eddy turbulence (spectral approach, inertial range, cascade, closure problem). In this context, a problem that attracted a lot of attention was that of gravity wave turbulence (which is an example of surface waves). This problem deals with four-wave resonant interactions: in this case, there are two inviscid invariants, energy and wave action. The first is characterized by a direct cascade and the second by an inverse cascade. The study carried out[15] by Zakharov and Filonenko (1966) (see also Zakharov and Filonenko, 1967) focused only on the energy spectrum. The authors obtained the exact solution as a power law associated with energy, but curiously they did not focus on the second solution and therefore did not immediately realize that it corresponded to a new type of cascade. Starting from a similar study (involving four-wave resonant interactions) on Langmuir wave turbulence by Zakharov (1967), in which the energy spectrum had also been obtained, Kaner and Yakovenko (1970) found the second exact solution corresponding to an inverse cascade of wave action. It is thus in the field of plasmas that the existence of a dual cascade was finally demonstrated in wave turbulence.[16]

A major difference between the two turbulence regimes is that, unlike (strong) eddy turbulence, (weak) wave turbulence theory is analytical (see Chapter 4). In this case, one can develop a uniform asymptotic theory and obtain the dynamic equations of the system and then, if they exist, its exact spectral solutions. It is then possible to provide analytical proof of the type of cascade (direct or inverse). It is also possible to prove the local character of turbulence (by a study of the convergence of integrals) and thus be in agreement with one of Kolmogorov's fundamental hypotheses. For this reason, exact nontrivial solutions of wave turbulence are called Kolmogorov–Zakharov spectra. There are several examples in wave turbulence where there is an inverse cascade of wave action; in Chapter 9 we present the case of gravitational wave turbulence (Galtier and Nazarenko, 2017). It is less common to obtain an inverse cascade in the case of three-wave resonant interactions. An example is given by rotating magnetohydrodynamic turbulence: the energy cascades directly and the hybrid helicity (a modified magnetic helicity) cascades inversely (Galtier, 2014).

To conclude this section, let us note that Robert Kraichnan and Vladimir Zakharov received the Dirac medal in 2003 for their contributions to the theory of turbulence, particularly the exact results and the predictions of inverse

15 Many other studies have been devoted to gravity wave turbulence. Chapter 4 discusses some of them.
16 The second exact solution corresponding to an inverse cascade of wave action for gravity waves was published by Zaslavskii and Zakharov (1982).

cascade, and for identifying classes of turbulence problems for which in-depth understanding has been achieved.

1.1.5 Emergence of Direct Numerical Simulation

From the 1970s, a new method for analyzing turbulence emerged: direct numerical simulation (Patterson and Orszag, 1971; Fox and Lilly, 1972). By direct, we mean the simulation of the fluid equations themselves and not a model of these equations. We have already cited as a model the EDQNM approximation used in hydrodynamics (Orszag, 1970); there is also the case of magnetohydrodynamics with the study of the inverse cascade of magnetic helicity (Pouquet et al., 1976). There are other models such as nonlinear diffusion models (Leith, 1967) or shell models (Biferale, 2003) – which we will briefly discuss in Chapter 3.

Since its beginnings, direct numerical simulation has made steady progress. It currently represents a means of studying turbulence in great detail; it is also an indispensable complement to experimental studies. It is impossible to summarize in a few lines the numerous results obtained in the field of numerical simulation. Let us simply point out that the regular increase in spatial resolution makes it possible to increase the Reynolds number and to describe increasingly fine structures (see Figure 1.2). It is interesting to compare the current situation with the first direct numerical simulations of incompressible three-dimensional hydrodynamic turbulence. For example, Orszag and Patterson (1972) used a spatial resolution of 64^3 and, as explained by the authors, each time step then required a computation time of 30 seconds! It is also interesting to note that the diffusion of knowledge takes some time: for example, the first direct numerical simulation of incompressible three-dimensional magnetohydrodynamic turbulence was realized by Pouquet and Patterson (1978) with a spatial resolution of 32^3. Nowadays, a standard direct numerical simulation of turbulence is generally performed with a pseudospectral code, in a periodic box and with a spatial resolution of about 2048^3 – the highest to date being $16\,384^3$ (Iyer et al., 2019). For more information on the subject, the reader can consult the review article of Alexakis and Biferale (2018), where numerous examples of direct numerical simulation are presented in the context of various turbulence studies.

1.1.6 Turbulence Today

In the history of sciences on turbulence, the early 1970s were a turning point. Very schematically, we can consider that the theory of turbulence was built during the years 1922–1972, a period during which the main concepts were introduced, allowing the first exact results to be obtained.[17] The books of Monin and

[17] The year 1922 can be used as a reference since it is this year that Richardson introduced the fundamental concept of energy cascade.