Finite Element and Finite Volume Methods for Heat Transfer and Fluid Dynamics

This book introduces the two most common numerical methods for heat transfer and fluid dynamics equations, using clear and accessible language. This unique approach covers all necessary mathematical preliminaries at the beginning of the book for the reader to sail smoothly through the chapters. Students will work step-by-step through the most common benchmark heat transfer and fluid dynamics problems, firmly grounding themselves in how the governing equations are discretized, how boundary conditions are imposed, and how the resulting algebraic equations are solved. Providing a detailed discussion of the discretization steps and time approximations, this graduate textbook has everything an instructor needs to prepare students for their exams and future careers. Each illustrative example shows students how to draw comparisons between the results obtained using the two numerical methods, and at the end of each chapter they can test and extend their understanding by working through the problems provided. A solutions manual is also available for instructors.

Professor J. N. Reddy is a Distinguished Professor, Regents Professor, and holder of the O'Donnell Foundation Chair IV in the Department of Mechanical Engineering at Texas A & M University. As the author of 24 textbooks and several hundred journal papers, and a highly cited researcher, Professor Reddy is internationally recognized for his research and education in applied and computational mechanics. He has won many major awards from professional societies (e.g., the S. P. Timoshenko Medal, the von Karman Medal, the von Neumann Medal, and the Gauss–Newton Medal). He is a member of the US National Academy of Engineering.

Professor N. K. Anand is a Regents Professor and James J. Cain '51 Professor III of Mechanical Engineering at Texas A & M University. He teaches and researches in the broad area of thermal sciences. Professor Anand is a recipient of the Association Former Students Distinguished Achievement Award in Teaching at college level. He is an ASME Fellow and was awarded the 2020 ASME Harry Potter Gold Medal. He developed and continues to teach a course in the application of finite volume techniques to heat transfer and fluid flow for three decades.

Dr. Pratanu Roy is a Staff Scientist at Lawrence Livermore National Laboratory (LLNL), California. Dr. Roy conducts research in computational fluid dynamics (finite volume method, multigrid methods), high-performance computing for CFD, transport in Carbon Capture and Storage (CCS), and turbulence modeling. Dr. Roy is the recipient of the 2015 American Rock Mechanics Association (ARMA) best paper award, the 2020 LLNL Physical and Life Sciences Directorate award, and in 2022 he was awarded National Nuclear Security Administration (NNSA) Defense Programs Awards of Excellence.

"I am delighted to recommend this textbook to beginners and early career researchers wanting to work in computational heat and fluid flow problems. This book is a useful tool for teaching postgraduate and senior undergraduate courses and will be an excellent addition to the bookshelves of senior researchers."

Perumal Nithiarasu, ***Swansea University***

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J. N. REDDY
Texas A & M University

N. K. ANAND
Texas A & M University

P. ROY
Lawrence Livermore National Laboratory, California

CAMBRIDGE
UNIVERSITY PRESS

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Shaftesbury Road, Cambridge CB2 8EA, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of Cambridge University Press & Assessment, a department of the University of Cambridge.

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www.cambridge.org
Information on this title: www.cambridge.org/highereducation/isbn/9781009275484
DOI: 10.1017/9781009275453

First published 2023

Printed in the United Kingdom by TJ Books Limited, Padstow Cornwall

A catalogue record for this publication is available from the British Library.

ISBN 978-1-009-27548-4 Hardback

Additional resources for this publication at www.cambridge.org/reddy-anand-roy.

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Preface

Our motivation for writing this book came from the need to introduce the two most popular numerical methods - the finite element method (FEM) and the finite volume method (FVM) - as techniques for solving differential equations arising in heat transfer and fluid dynamics problems. As experienced teachers and users of FEM and FVM courses, we felt well-placed to help students acquire a unified and practical understanding of these methods, which will prepare them for future employment and study.

The book is designed for senior undergraduate and first-year graduate students who have had courses in linear algebra, differential equations, and undergraduate level heat transfer and fluid mechanics, and some programming experience. However, additional courses (or exposure to the topics covered) in heat transfer and fluid mechanics should make the student feel more comfortable with the physical examples discussed in the book.

In both the FEM and FVM, the geometric region of the problem (on which a differential equation is to be solved) is represented as a collection of subregions, called a mesh or a grid. In the FEM (and also in the so-called "control volume finite element method"[1]), these subregions are called finite elements, which have associated interpolation functions defined on each element. Then a second mesh of subregions is overlaid on the first mesh, with the objective of developing the discretized equations. In the FEM, the second mesh of finite elements is used to define the interpolation of the dependent unknown as well as to derive the finite element equations using a method of approximation (e.g., weak-form Galerkin method, least-squares method, and so on) over the element. When the two meshes are exactly the same in the FEM, it is called an isoparametric formulation.

In the FVM, the domain is divided into control volumes and typically a node is deployed at the geometric center of each control volume. Nodes are the locations at which unknowns or dependent variables are calculated. Discretization equations at each node are generated by discretizing flux balance (namely, mass, momentum, or energy) equations over each control volume that surrounds the node. In broad terms, the discretization equations at each node in FVM are

[1] The control volume finite element method (CVFEM) is a misnomer; it is nothing to do with the FEM, other than that CVFEM also uses interpolation functions used in the FEM as the approximation functions. The interpolation theory predates the FEM.

obtained by a combination of numerical integration and differentiation of model differential equations over a control volume. In the FEM, the discretized equations over a finite element involve nodal values of that element only, whereas in the FVM, the discretized equations for each control volume involves the nodal values of control volumes adjacent to the control volume under consideration. Thus, the two methods are different in deriving the discretized equations of the problem: The FEM uses a weighted-integral statement to derive the discretized equations over an element and "assembles" the element equations to obtain the discretized equations of the whole domain. It is fair to characterize the FEM as an integral method, the finite difference method (FDM) as a method in which the derivatives are represented with suitable difference formulas, and the FVM as a combination of the two approaches.

The book is broadly divided into three parts. Part I deals with background needed for the later chapters of the book, Part II is concerned with the finite element method, and Part III is devoted to the finite volume method. This division is useful to the reader in skipping any of the parts, as they are fairly independent of each other.

In the present study of the FEM and FVM, advanced mathematics is intentionally avoided in the interest of simplicity. However, a minimum of mathematical machinery that seemed necessary is included in Chapter 1. In Chapter 2, a review of the governing equations of a continuous medium, with focus on heat transfer and fluid mechanics, is presented. It is here background from a continuum mechanics course or undergraduate-level fluid mechanics and heat transfer courses prove to be helpful. Chapter 3 is devoted to a discussion of various methods for solving linear algebraic equations.

In introducing the FEM in Chapter 4 for steady-state heat transfer, the traditional solid mechanics approach is avoided in favor of the "differential equation" approach, which has broader interpretations than a single special case. Since a large number of physical problems are described by second-order differential equations, they are used as model equations in introducing the method. Considerable attention is devoted to the finite element formulation, the derivation of interpolation functions, and the solution of problems to illustrate the main features of the FEM. The FEM for unsteady heat transfer is introduced in Chapter 5 using the same model equations. Weak forms, semidiscrete finite element models, and time approximations are discussed in detail, and concepts of explicit and implicit formulations are examined. Several numerical examples are presented to illustrate how the numerical stability of the forward difference scheme is connected to the critical time step. Chapter 6 is dedicated to the flows of viscous incompressible flows in two dimensions. The velocity–pressure and penalty function formulations are introduced and associated finite element models are presented for Stokes equations (where the convective terms are neglected) and the Navier–Stokes equations. Several numerical examples of steady as well as unsteady problems of viscous flows are presented.

In Chapter 7, we introduce the FVM as applied to steady as well as transient heat transfer in one and two dimensions. Chapter 8 is devoted to the study of advection–diffusion equations in one and two dimensions. Chapter 9 deals with viscous incompressible flows using the FVM. Finally, Chapter 10 is concerned with some advanced topics, including periodically fully developed flows and heat transfer, natural convection, and multigrid techniques. Both FEM and FVM are used in solving a number of benchmark problems. Emphasis is placed on the modeling issues, such as the selection of mesh, imposition of boundary conditions, solution of equations, and interpretation of the results.

At the end of each chapter, especially in the earlier chapters, a number of exercise problems are included to test and extend the understanding of the concepts discussed. A solutions manual of most problem has been prepared, and it is available through the publisher to teachers who adopt the book in their courses. The computer programs used to numerically solve the problems discussed in the examples of various chapters are not included in this book owing to space limitations and proprietary nature of the programs.

Professor Reddy and Professor Anand thank Texas A&M University for providing them with an academic home and the ecosystem that led to the writing of this book. They also thank Mr. Mitch Wittneben for assisting with the software for the book writing. Contributions of several graduate students and post-docs in proof-reading various parts of the manuscript are gratefully acknowledged. The first two authors thank the post-doctoral fellow Praneeth Nampally and graduate students Buyng-Hee Choi, Daniel Orea, Rey Chavez, and Khoi Ngo for assisting with the proof reading of the manuscript. Dr. Roy thanks his mother and colleagues at Lawrence Livermore National Laboratory for encouraging and supporting his participation in this book project. Dr Roy's contribution was performed under the auspices of the U.S. Department of Energy by Lawrence Livermore National Laboratory under Contract DE-AC52-07NA27344. The authors are also grateful to the staff at Cambridge University Press for their help and support during the production of this book. Finally, authors thank their spouses for their understanding and support while the authors were occupied with the writing of this book.

To our respective wives,

Aruna Reddy, Veena Anand, and *Koly Sengupta*

Each of the authors is grateful to his wife for the love and support while he was occupied with the preparation of this book.

OM Sahana Vavatu
Sahanau Bhunaktu
Saha Viryam Karavavahai
Tejasvi Navaditamastu
Ma Vidvishavahai
OM Shaantih Shaantih Shaantih
(from **Rigveda**)

Meaning of the above "shloka" is as follows ("two" refers to the Teacher and Student):

OM, Together may we two move
Together may we two relish
Together may we perform (our studies) with vigour
May what has been studied by us be filled with the brilliance (of Understanding, leading to Knowledge); *May it not give rise to hostility* (due to lack of understanding)
OM Peace, Peace, Peace

Symbols

The symbols that are used in the book for various important quantities are defined in the following.

Symbol	Meaning
$\mathbf{a}$	Acceleration vector, $\frac{D\mathbf{v}}{Dt}$
a_{ij}	Coefficients of matrix $[A] = \mathbf{A}$
c_v, c_p	Specific heat at constant volume and pressure, respectively
d	Diameter
ds, dS	Surface elements
dA	Area element $(= dxdy)$
dv	Volume element $(= dxdydz)$
$\mathbf{D}$	Symmetric part of the velocity gradient tensor; that is, $\mathbf{D} = \frac{1}{2}\left[(\boldsymbol{\nabla}\mathbf{v})^{\mathrm{T}} + \boldsymbol{\nabla}\mathbf{v}\right]$
D_{ij}	Rectangular Cartesian components of $\mathbf{D}$
D/Dt	Material time derivative, $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}$
$\mathbf{e}_i$	Basis vector in the x_i-direction
$(\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_z)$	Basis vectors in the (r, θ, z) system
$(\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z)$	Basis vectors in the (x, y, z) system
$(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$	Basis vectors in the (x_1, x_2, x_3) system
$\mathbf{f}$	Body force vector
f_x, f_y, f_z	Body force components in the x-, y-, and z-directions
g	Internal heat generation per unit volume
g_f	Acceleration due to gravity
$\mathbf{I}$	Unit second-order tensor
$\mathbf{g}_f$	Acceleration due to gravity vector
$\mathbf{I}$	Unit second-order tensor
J	Determinant of $\mathbf{J}$ (Jacobian)
$\mathbf{J}$	Jacobian (of transformation) matrix

Symbol	Meaning
k	Thermal conductivity
$\mathbf{k}$	Thermal conductivity tensor
K	Kinetic energy
L	Length
$\hat{\mathbf{n}}$	Unit normal vector in the current configuration
n_i	ith component of the unit normal vector $\hat{\mathbf{n}}$
(n_x, n_y, n_z)	Components of the unit normal vector $\hat{\mathbf{n}}$
P	Hydrostatic pressure; perimeter
q_n	Heat flux normal to the boundary, $q_n = \boldsymbol{\nabla} \cdot \hat{\mathbf{n}}$
$\mathbf{q}$	Heat flux vector; diffusion flux
Q	Heat; mass flow rate; volume rate of flow
r	Radial coordinate in the cylindrical polar system; $r = \|\mathbf{r}\|$
$\mathbf{r}$	Position vector in cylindrical coordinates, $\mathbf{x}$
(r, θ, z)	Cylindrical coordinate system
R	Residual in the approximation; radius
t	Time
$\mathbf{t}$	Stress vector; traction vector
$\mathbf{t}_i$	Stress vector on x_i-plane, $\mathbf{t}_i = \sigma_{ij}\hat{\mathbf{e}}_j$
T	Temperature
v	Velocity, $v = \|\mathbf{v}\|$
(v_1, v_2, v_3)	Components of velocity vector $\mathbf{v}$ in (x_1, x_2, x_3) system
(v_r, v_θ, v_z)	Components of velocity vector $\mathbf{v}$ in (r, θ, z) system
$\mathbf{v}$	Velocity vector, $\mathbf{v} = \frac{D\mathbf{x}}{Dt}$
$\mathbf{v}_n$	Velocity vector normal to the plane (whose normal is $\hat{\mathbf{n}}$)
$\mathbf{x}$	Position vector in the current configuration
(x, y, z)	Rectangular Cartesian coordinates
(x_1, x_2, x_3)	Rectangular Cartesian coordinates

Other Symbols

Symbol	Meaning
$\boldsymbol{\nabla}$	Gradient operator with respect to $\mathbf{x}$
∇^2	Laplace operator, $\nabla^2 = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$
∇^4	Biharmonic operator, $\nabla^4 = \nabla^2 \nabla^2$
[ ]	Matrix of components of the enclosed tensor
{ }	Column of components of the enclosed vector
$\cdot$	Symbol for the dot product or scalar product
$\times$	Symbol for the cross product or vector product

Greek Symbols

Symbol	Meaning
α	Angle; coefficient of thermal expansion; relaxation factor
β	Heat transfer coefficient
γ	Penalty parameter
Γ	Total boundary
δ	Dirac delta; variational symbol
δ_{ij}	Components of the unit tensor, $\mathbf{I}$ (Kronecker delta)
ε_{ijk}	Alternating symbol
ζ	Natural (normalized) coordinate
η	Natural (normalized) coordinate
θ	Angular coordinate in the cylindrical and spherical coordinate systems; angle; absolute temperature
λ	Lagrange multiplier
μ	Viscosity
ξ	Natural (normalized) coordinate
ρ	Mass density
$\boldsymbol{\sigma}$	Stress tensor
σ_{ij}	Components of the stress tensor in the rectangular coordinate system (x_1, x_2, x_3)
$\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta}, \ldots$	Components of the stress tensor $\boldsymbol{\sigma}$ in the cylindrical coordinate system (r, θ, z)
τ	Shear stress
$\boldsymbol{\tau}$	Viscous stress tensor
ϕ	A typical scalar function; velocity potential; angular coordinate in the spherical coordinate system
ϕ_i	A generic variable
ψ	Stream function
ψ_i	Lagrange interpolation functions
ω	Angular velocity
Ω	Domain of a problem
$\boldsymbol{\Omega}$	Spin tensor or skew symmetric part of the velocity gradient tensor, $(\boldsymbol{\nabla}\mathbf{v})^{\mathrm{T}}$; that is $\boldsymbol{\Omega} = \frac{1}{2}\left[(\boldsymbol{\nabla}\mathbf{v})^{\mathrm{T}} - \boldsymbol{\nabla}\mathbf{v}\right]$
ω_i	Components of vorticity vector $\boldsymbol{\omega}$ in the rectangular coordinate system (x_1, x_2, x_3)
$\omega_x, \omega_y, \omega_z$	Components of vorticity vector $\boldsymbol{\omega}$ in the rectangular coordinate system (x, y, z)