

Analytic Projective Geometry

Projective geometry is the geometry of vision, and this book introduces students to this beautiful subject from an analytic perspective, emphasising its close relationship with linear algebra and the central role of symmetry. Starting with elementary and familiar geometry over real numbers, readers will soon build upon that knowledge via geometric pathways and journey on to deep and interesting corners of the subject. Through a projective approach to geometry, readers will discover connections between seemingly distant (and ancient) results in Euclidean geometry. In mixing results from the past 100 years with the history of the field, this text is one of the most comprehensive surveys of the subject and an invaluable reference for undergraduate and beginning graduate students learning classic geometry, as well as young researchers in computer graphics. Students will also appreciate the worked examples and diagrams throughout.

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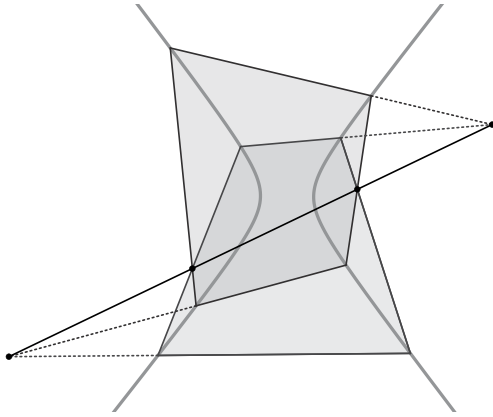
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“This book provides a lively and lovely perspective on real projective spaces, combining art, history, groups, and elegant proofs.”

– William M. Kantor

“This book is a celebration of the projective viewpoint of geometry. It gradually introduces the reader to the subject, and the arguments are presented in a way that highlights the power of projective thinking in geometry. The reader surprisingly discovers not only that Euclidean and related theorems can be realised as derivatives of projective results, but there are also unnoticed connections between results from ancient times. The treatise also contains a large number of exercises and is dotted with worked examples, which help the reader to appreciate and deeply understand the arguments they refer to. In my opinion this is a book that will definitely change the way we look at the Euclidean and projective analytic geometry.”

– Alessandro Siciliano, Università degli Studi della Basilicata



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Preface

Projective geometry is the geometry of vision. Yet Arthur Cayley saw that it is *all* geometry. The mathematical historian Morris Kline called it the ‘science born of art’, and the very early history of its development from that origin is documented in the book *The Geometry of an Art* by the later mathematical historian Kirsti Andersen. Some of those developments (and some later ones) appear in Chapters 11 and 12, and what Cayley meant is explained in Chapters 7 through 10. Felix Klein also advocated for the centrality of projective geometry, but is better known for bringing out the central role of symmetry in geometry in his *Erlangen Programme*. Our treatment of most of the topics in this book emphasises this central role of symmetry, through the prominent place we assign groups, and we also explain Klein’s view on transformation geometry in Chapter 9. Moreover, this whole subject has a close relationship with linear algebra, and this underpins our treatment. What Jürgen Richter-Gebert calls ‘the beauty that lies in the rich interplay of geometric structures and their algebraic counterparts’ is a recurring theme of our book. Finally, we try to illustrate some of the advantages gained by taking a projective approach to geometry in Chapter 10, where we obtain connections between seemingly distant (and ancient) results in Euclidean geometry via the perspective hard-won in the earlier chapters.

This book is an introduction to projective geometry, and our coordinates are mostly over the real numbers. However, there is advanced and novel material for the practician. Chapter 6 examines one of the leitmotifs of this book – *involutions* and their role in projective geometry. This is taken much further in Section 10.2, where we begin with an old result of Pappus, and explore the more modern theorems of Ferrers, Jeřábek, Lehmer & Daus, Gardner & Gale, Robson & Strange, and their astounding synergy.

We find that an approach that teaches the subject conceptually while also sketching its development resonates with us as teachers and authors, and also hope that it will find sympathetic vibrations in students and readers.