

# Index

- absolutely continuous
  - function, 117
  - measure, 423
  - set function, 443
- adjoint of an operator, 338
- algebra, 287
- almost everywhere, 14
- amenable group, 158
- analytic function, 304
- annihilator
  - of a positive linear functional, 503
  - of a subspace, 131
- approximate unit, 382
- Arzela-Ascoli Theorem, 198
- atom of a measure, 512
- atomless measure space, 512
  
- Baire Category Theorem, 161
- Baire set, 407
- balanced set, 273
- Banach algebra, 288
- Banach Contraction Principle, 68
- Banach Limit, 137
- Banach space, 37
- Banach-Alaoglu Theorem, 245
- Banach-Steinhaus Theorem, 164
- basis
  - Faber-Schauder, 206
  - Haar system, 190
  - Hamel, 2
  - orthonormal, 82
  - Schauder, 183
  - standard, 3
  - standard orthonormal basis, 90
- basis constant, 187
  
- Bessel's Inequality, 85
- Best Approximation Property, 73
- Borel
  - measurable function, 8, 459
  - $\sigma$ -algebra, 7
  - set, 7
- boundary of a set, 314
- bounded below, 312
- Bounded Inverse Theorem, 172
- bounded pointwise convergence, 461
- bounded variation function, 442
  
- $C^*$ -algebra, 346
- $C^*$ -identity, 346
- $C^*$ -subalgebra, 348
  - generated by a set, 348
- Calkin algebra, 384
- Carathéodory's Extension Theorem, 410
- Cauchy-Schwarz Inequality, 20, 95, 405
- Cesáro mean, 203
- characteristic function, 8
- cl-open set, 410
- Closed Graph Theorem, 176
- closed half-space, 283
- codimension, 6
- commutant of a subalgebra, 475
- commuting normal operators, 517
- compactly supported functions, 66
- complement
  - algebraic, 208
  - topological, 208
- complemented subspace, 208
- completely continuous operator, 278

- completion  
   of a normed linear space, 62  
   of an inner product space, 99  
 complex-measurable function, 8  
 complexification of a Hilbert space, 97  
 conjugate exponent, 24, 111  
 conjugate-linear isometry, 79  
 contraction, 68  
 convex  
   function, 189  
   set, 64, 73  
 convex hull of a set, 230, 274  
 coordinate functional, 183  
 countable additivity, 9, 416  
 cyclic  
   representation, 459  
   vector, 453, 459  
  
 diagonalizable operator, 445, 447, 453  
 dimension  
   of a Hilbert space, 88  
   vector space, 4  
 direct sum  
   of Hilbert spaces, 77, 98, 508  
   of operators, 508  
   of representations, 508  
 Dirichlet kernel, 166  
 disc algebra, 326  
 division ring, 305  
 Dominated Convergence Theorem, 14  
 double centralizer, 394  
 double dual, 139  
 dual space, 32  
 dyadic rational, 189  
 Dynkin  $\pi - \lambda$  Theorem, 408  
  
 Eberlein's Theorem, 234  
 Eberlein-Šmulian Theorem, 250  
 Egoroff's Theorem, 217  
 eigenvalue, 311  
 eigenvector, 311  
 equicontinuous family of functions,  
   197  
 equivalent norms, 46  
 essential range of a function, 451  
 essential supremum, 26  
 essentially bounded function, 25  
 exponential, 299, 500  
 extension  
   of a linear functional, 81, 123  
   of a linear operator, 80  
  
 extremally disconnected space, 409  
 extreme point, 64, 259  
  
 face  
   of a convex set, 260  
   proper, 260  
 Fatou's Lemma, 13  
 Fejér kernel, 203  
 Fejér's Theorem, 204  
 finitely additive set function, 441  
 First Isomorphism Theorem  
   for Banach algebras, 295  
   for Banach spaces, 172  
   for  $C^*$ -algebras, 385  
   for normed linear spaces, 56  
   for vector spaces, 6  
 Fourier  
   coefficient, 94  
   cosine coefficient, 106  
   expansion, 88, 94  
   series of an  $L^1$  function, 179  
   series of an  $L^2$  function, 94  
   sine coefficient, 106  
   transform, 90  
 Fredholm Alternative, 322  
 Fubini-Tonelli Theorem, 15  
 Fuglede's Theorem, 498  
 Fuglede-Putnam Theorem, 499  
 functional  
   Minkowski, 219  
   sublinear, 218  
 functional calculus  
   Borel, 469  
   continuous, 375  
 Fundamental Theorem of Calculus, 118  
  
 Gelfand representation, 361  
 Gelfand spectrum, 357  
 Gelfand transform, 361  
 Gelfand-Mazur Theorem, 306  
 Gelfand-Naimark Theorem, 365, 510  
 Gelfand-Naimark-Segal (GNS) construction,  
   505  
 Goldstine's Theorem, 270  
 Gram determinant, 96  
 Gram-Schmidt Orthogonalization, 84  
 graph norm, 176  
 graph of a function, 175  
  
 Hölder's Inequality, 23  
 Hahn-Banach Separation Theorem, 222

- complex Case, 224, 258
  - for locally convex spaces, 257
  - for topological vector spaces, 253
- Hahn-Banach Theorem
  - for Hilbert spaces, 81
  - General case, 128
  - Real case, 126
- Hardy space, 106
- Heine–Borel Theorem, 48
- Helley’s Theorem, 233, 250
- Hellinger-Toeplitz Theorem, 203
- Hermite polynomial, 103
- Hilbert space, 37
- hole, 371
- homomorphism
  - of Banach algebras, 293
  - of  $C^*$ -algebras, 350
- hyperplane, 6
  - affine, 221
- ideal, 290
  - essential, 395
  - in a  $C^*$ -algebra, 381
  - left, 290
  - maximal, 290
  - right, 290
- idempotent element, 332
- inner product, 19
  - Euclidean, 21
- inner product space, 19
- integrable function, 12
- integral operator, 36
- invariant subspace, 517
- inverse of an element, 298
- invertible element, 298
- involution, 346
- isomorphism
  - isometric, 49
  - of Banach algebras, 293
  - of  $C^*$ -algebras, 350
  - of Hilbert spaces, 90
  - topological, 49
  - vector space, 5
- Kantorovič Extension Theorem, 429
- kernel
  - of a linear operator, 5
  - of an integral operator, 36
  - reproducing, 107
- Krein-Milman Theorem, 263
- Lebesgue’s Differentiation Theorem, 117
- Legendre polynomial, 85
- linear combination, 2
- linear functional, 4
  - multiplicative, 357
  - positive, 67
  - self-adjoint, 428
  - SOT-continuous, 473
  - WOT-continuous, 473
- linear operator, 4
  - bounded, 29
  - compact, 191
  - continuous, 29
  - finite rank, 191
  - isometry, 90, 343
  - normal, 341
  - order preserving, 402
  - positive, 402
  - self-adjoint, 341
  - unitary, 90, 343
- linear transformation, 4
- linearly independent, 2
- locally compact space, 51
- locally convex space, 253, 472
- lower semi-continuous function, 275
- Lusin’s Theorem, 427, 514
- MASA, 478
- matrix algebra, 510
- matrix of an operator, 445
- Mazur’s Theorem, 230
- measurable function, 8
- measurable set, 7, 11
- measurable space, 7
- measure
  - Baire, 407, 424
  - Borel, 10
  - complex, 416
  - counting, 10
  - Dirac, 10
  - finite, 10
  - finitely additive, 158
  - Lebesgue, 10
  - negative variation of, 531
  - positive, 9
  - positive variation of, 531
  - product, 15
  - real, 416
  - regular, 10, 427, 436
  - $\sigma$ -finite, 10
  - signed, 416

- spectral, 483
- translation invariant, 10
- measure space, 9
- metric induced by a norm, 17
- Milman-Pettis Theorem, 270
- Minkowski's Inequality, 25
- modulus, 401
- Monotone Convergence Theorem, 13, 462
- multiplication operator, 294, 449
- multiplier algebra, 394
- $n$ -simplex, 259
- net, 86
- Neumann series, 299
- nilpotent operator, 306
- norm, 17
  - 1-norm, 18
  - induced by an inner product, 20
  - $p$ -norm, 27
  - supremum, 18, 19
- norm-norm continuous operator, 276
- norm-closed set, 226
- norm-compact set, 226
- norm-open set, 226
- norm-weak continuous operator, 276
- normal element, 353
- normed algebra, 288
- normed linear space, 17
- nowhere dense set, 162
- nullity, 7
- open map, 54, 172
- Open Mapping Theorem, 172
- operator norm, 32
- orthogonal
  - elements, 72
  - projection, 75
  - set, 82
- orthonormal set, 82
- $p$ -integrable function, 22
- $p$ -summable function, 89
- Parallelogram law, 72
- Parseval's identity, 88, 94
- partial isometry, 489
  - final space of, 489
  - initial space of, 489
- partially ordered vector space, 402
- Polar decomposition
  - of a measure, 423
  - of an operator, 490
- Polarization Identity, 72
- positive element
  - in  $C(X)$ , 67
  - in a  $C^*$ -algebra, 397
- Principle of Uniform Boundedness, 164
- projection, 376
- Prokhorov's Theorem, 441
- Pythagoras' Theorem, 72
- quasinilpotent element, 311
- quotient map, 6
- quotient space
  - of a vector space, 6
- Radon-Nikodym derivative, 527
- Radon-Nikodym Theorem, 527
- range of a linear operator, 5
- rank of an operator, 7
- Rank-Nullity Theorem, 6
- real-measurable function, 8
- reducing subspace, 456
- reflexive space, 140
- representation
  - direct sum of, 508
  - equivalent, 503
  - faithful, 510
  - Gelfand-Naimark-Segal (GNS), 503
  - irreducible, 516
  - left regular, 294
  - non-degenerate, 459
  - of a  $C^*$ -algebra, 459
  - $\sigma$ -normal, 461
  - universal, 510
- reproducing kernel Hilbert space, 107
- resolution of the identity, 483
- resolvent set, 302
- Riemann-Lebesgue Lemma, 86
- Riemann-Lebesgue Lemma, 94, 180
- Riesz Representation Theorem, 79, 121
- Riesz' Lemma, 50
- Riesz-Fischer Theorem, 41, 94
- Riesz-Markov-Kakutani Theorem, 407
- right shift operator, 90
- $\sigma$ -algebra, 7
  - Baire, 407
  - Borel, 7
  - generated by a set, 7
- Schur's Theorem, 214
- self-adjoint element, 353
- self-adjoint subset, 348

- semi-inner product, 95
- seminorm, 53
- separable space, 44
- separate points, 130, 177
- sequence
  - almost convergent, 156
  - periodic, 157
  - square-summable, 21
  - strongly convergent, 460
  - weakly Cauchy, 278
  - weakly convergent, 211, 460
- series
  - absolutely convergent, 40
  - convergent, 40
- sesquilinear form, 386
- signum (sign) function, 112
- simple function, 8
- Singular Value Decomposition, 511
- Sobolev space, 61
- Spectral Mapping Theorem, 307, 376
- Spectral Permanence Theorem, 372
- spectral radius, 306
- Spectral Radius Formula, 308
- Spectral Theorem
  - for compact operators, 447, 495
  - general case, 458
  - on a finite dimensional space, 447
  - special case, 454
  - spectral measure, 488
- spectral value, 302
- spectrum
  - approximate point spectrum, 312
  - essential, 393
  - of an element, 302, 353
  - of an operator, 311
  - point spectrum, 311
- square root, 401
- state
  - faithful, 519
  - on a  $C^*$ -algebra, 509
  - pure, 518
- step function, 120
- Stone-Čech compactification, 413
- Stone-Weierstrass Theorem, 524
- strictly convex function, 267
- strictly convex space, 151, 265
- strong operator topology (SOT), 470
- strongly convergent sequence, 211
- strongly open set, 226
- subalgebra, 290
- submultiplicativity of the norm, 288
- support of a measure, 437
- topological group, 300
- topological vector space, 228
- total variation of a measure, 418
- trace, 520
- transpose of an operator, 144
- Triangle inequality, 17
- trigonometric polynomial, 92
- Tychonoff's Theorem, 243
- uniformly convex space, 265
- unit ball
  - closed, 29
  - open, 29
- unit sphere, 29
- unital Banach algebra, 295
- unital subalgebra, 348
- unitization
  - of a Banach algebra, 329
  - of a  $C^*$ -algebra, 351
- unitarily equivalent operators, 452
- unitary element, 376
- unitary group, 500
- upper semi-continuity of the spectrum, 331
- Urysohn's Lemma, 42, 296
- vanish at infinity, 66
- vector space, 1
- Vigier's Theorem, 482
- Volterra integral equation, 68
- von Neumann algebra, 475
- weak closure of a set, 227
- weak operator topology (WOT), 470
- weak topology, 225, 226
- weak- $*$  convergent sequence, 236
- weak- $*$  topology, 238
- weak-norm continuous operator, 276
- weak-weak continuous operator, 276
- weakly bounded set, 200
- weakly closed set, 226
- weakly compact set, 226
- weakly convergent sequence, 211
- weakly open set, 226
- weakly sequentially compact, 232
- weakly sequentially complete, 278
- Weierstrass' Approximation Theorem, 523
- weighted shift operator, 334
- Zorn's Lemma, 3