Functional Analysis

Functional Analysis is a part of mathematics that deals with linear spaces equipped with a topology. The subject began with the work of Fredholm, Hilbert, Banach and others in the early 20th century. They developed an algebraic/topological framework which could be used to address a variety of questions in analysis. The subject immediately saw connections to abstract algebra, partial differential equations, geometry and much more.

This book is meant to introduce the reader to functional analysis. The first half of the book will cover the basic material that is taught in Masters programs across the world and prove all the major theorems in great detail. The second half of the book will focus on operators on a Hilbert space and is built around the proof of the spectral theorem – a central result in the subject that ties together traditional functional analysis with the modern theory of operator algebras.

The book aims to provide an accessible, interesting and readable introduction to the subject. It will also take the reader a little further than most courses do by introducing them to the language of operator algebras. This will help future researchers by giving them a jumping off point as they dive into deeper books on the subject.

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Preface

This book is an attempt to emulate the classroom learning experience. It seems appropriate in a world where online education has become par for the course and the student does not always have access to a teacher who can help fill in the blanks. As a result, the book is thorough (sometimes to a fault) and somewhat more conversational than most others of its ilk.

The classroom is a place where one often engages in free-wheeling discussions that cut across disciplines. The subject of Functional Analysis, which lies at the confluence of modern analysis, algebra and topology, seems well-placed to transfer such discussions to the written word. It seamlessly mixes ideas from these different subjects, is widely applicable, and is therefore appealing to a broad spectrum of people. My hope is to present an introduction to the subject that is useful to everyone, regardless of their tastes.

The book is intended to be used for a year-long course in Functional Analysis aimed at Master's or Ph.D. students. After a short review in Chapter 1, Chapters 2–6 constitute the core of the subject. Here, one proves the Hahn-Banach theorems, the consequences of the Baire Category theorem, and the Banach-Alaoglu and Krein-Milman Theorems. Barring a few specialized topics, these chapters may be taught in a single semester.

The second half of the book (Chapter 7–10) is a little more advanced, and is meant to be taught in the second semester as an introduction to the theory of Operator Algebras. Ostensibly, the goal is to prove the Spectral Theorem for Normal Operators on a Hilbert space. However, I have chosen to take the scenic route, introducing as much operator algebra theory as I can given the time constraints. Perhaps the most egregious detour is in Chapter 9, where one encounters a proof of the Riesz-Markov-Kakutani theorem (due to Garling) that uses C*-algebra theory. I hope that such discussions will encourage students to look further into this fascinating subject.

A word on the exercises: there are plenty of them at the end of each chapter. Many are there to complement the results proved in the text, while others are there to allow students to practice using these results. At the end of each chapter, there is a list of problems that are somewhat tangential to the topic at hand (for example, Reproducing Kernel Hilbert spaces, Amenable Groups, etc.). These are meant to introduce students to interesting questions and avenues of research. As such, they may be used as jumping-off points for short projects. I have also mentioned some books and articles one might use for further investigation.

Finally, I should say that this book is far from comprehensive. Indeed, it would be impossible to do justice to a subject that is as vast and varied as Functional Analysis. I hope, though, that the book will give the reader enough tools to understand more advanced texts with confidence.

The book was conceived and written during the interminable lockdowns necessitated by the COVID-19 pandemic. During this time, Namrata and Nayan have had to put up with me, my temper, and my all-pervading supply of paper. I want to thank them for their patience and good humour through all of it. I also want to thank Vidya, Viswa, Kartik, Vidya (Sr.) and all the others in the extended family whose affection and support over the years means so much. Finally, I owe the greatest debt of gratitude to my parents. It is thanks to their sacrifices that I am able to live a comfortable and happy life today. For this (and so much more), this book is fondly dedicated to them.

Notation

Throughout the book, I plan to use some notational conventions which I have described below. Apart from these, a variety of symbols are used *locally*, which will be defined as and when they appear.

Notation	What it represents
$\mathbb{C}$	The set of complex numbers
$\mathbb{N}$	The set of natural numbers
$\mathbb{Q}$	The set of rational numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{Z}$	The set of integers
μ, ν, λ	Measures
$\mathfrak{M}, \mathfrak{N}$	σ-algebras
$\mathfrak{L}$	The Lebesgue σ-algebra
$\mathfrak{B}_X$	The Borel σ-algebra on X
$\mathfrak{A}_X$	The Baire σ-algebra on X
$\mathbb{K}$	The base field for a vector space
$\alpha, \beta, \ldots$	Scalars in a field
$x, y, z, \ldots$	Elements of a vector space
$f, g, h, \ldots$	Scalar-valued functions, and elements of function spaces
$\mathbf{A}, \mathbf{B}, \mathbf{C}$	Banach or C*-algebras
$\mathbf{I}, \mathbf{J}$	Ideals in a Banach or C*-algebra
$\mathbf{E}, \mathbf{F}, \mathbf{W}, \mathbf{N}$	Normed linear spaces
$\mathbf{H}, \mathbf{K}, \mathbf{M}$	Hilbert spaces
φ, ψ	(Bounded) linear functionals
$T, S, \ldots$	(Bounded) linear operators
Φ, Ψ	Homomorphisms between Banach or C*-algebras

Notation	What it represents
$\mathcal{B}(\mathbf{E})$	The set of bounded linear operators on **E**
$\mathcal{K}(\mathbf{E})$	The set of compact linear operators on **E**
$\mathcal{F}(\mathbf{E})$	The set of bounded finite rank operators on **E**
$\triangleleft$	Ideal (used as $\mathbf{I} \triangleleft \mathbf{A}$)
$\mathbf{1}$	Constant function 1
$\xrightarrow{w}$	Weak convergence; also used for WOT-convergence
$\xrightarrow{s}$	Strong (norm) convergence; also used for SOT-convergence
$\xrightarrow{w^*}$	Weak-$*$ convergence
$\xrightarrow{bp}$	Bounded pointwise convergence

Among the exercises, those marked with the (⚒) symbol may be treated as an extension of the text, and must be solved by any serious student of the subject.