Chapter 1

Preliminaries

1.1 Review of Linear Algebra

Linear algebra is primarily the study of finite dimensional vector spaces and linear transformations between them. In a first course, one encounters the fact that every such vector space has a basis, that linear transformations may be associated to matrices, and one finally understands why matrix multiplication looks so clumsy. This section sets out to relive those glory days.

1.1.1 DEFINITION A ***vector space*** over a field $\mathbb{K}$ is a set $\mathbf{E}$ together with two operations $a\colon \mathbf{E} \times \mathbf{E} \to \mathbf{E}$ (vector addition) and $s\colon \mathbb{K} \times \mathbf{E} \to \mathbf{E}$ (scalar multiplication) written as $a(x,y) = x+y$ and $s(\alpha,x) = \alpha x$, which satisfy the following properties.

(a) $(\mathbf{E}, +)$ is an abelian group, whose identity is denoted by 0. If $x \in \mathbf{E}$, we write $(-x)$ for the inverse of x.

(b) If $1 \in \mathbb{K}$ denotes the multiplicative identity, then $1x = x$ for all $x \in \mathbf{E}$.

(c) $\alpha(\beta x) = (\alpha\beta)x$ for all $\alpha, \beta \in \mathbb{K}$ and $x \in \mathbf{E}$.

(d) $\alpha(x+y) = \alpha x + \alpha y$ for all $\alpha \in \mathbb{K}$ and all $x, y \in \mathbf{E}$.

(e) $(\alpha+\beta)x = \alpha x + \beta x$ for all $\alpha, \beta \in \mathbb{K}$ and $x \in \mathbf{E}$.

Note. For reasons we will discuss later, all vector spaces in this book will be over $\mathbb{R}$ or $\mathbb{C}$ (denoted by $\mathbb{K}$ when we do not wish to specify which). Furthermore, for the sake of sanity, we will always assume that our vector spaces are *non-zero*.

1.1.2 DEFINITION Let $\mathbf{E}$ be a vector space over $\mathbb{K}$ and let $S = \{x_1, x_2, \ldots, x_n\}$ be a finite set of vectors in $\mathbf{E}$.

(i) A ***linear combination*** of these vectors is an expression of the form $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$, where $\alpha_i \in \mathbb{K}$ for all $1 \leq i \leq n$.

(ii) The set S is said to be ***linearly dependent*** if there is some vector in S which can be expressed as a linear combination of the remaining vectors. Equivalently, S is linearly dependent if there are scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $\mathbb{K}$, not all of which are zero, such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 0. \tag{1.1}$$

(iii) The set S is said to be ***linearly independent*** if it is not linearly dependent. Equivalently, S is linearly independent if, whenever $\alpha_1, \alpha_2, \ldots, \alpha_n$ are scalars such that Equation 1.1 holds, then each α_i is forced to be zero.

1.1.3 DEFINITION A ***Hamel basis*** for a vector space $\mathbf{E}$ is a set $\Lambda \subset \mathbf{E}$ such that every element of $\mathbf{E}$ can be expressed uniquely as a finite linear combination of elements in Λ.

The word 'finite' is crucial in the above definition. An infinite sum is necessarily defined as a *limit* of partial sums and therefore only makes sense in a vector space that is equipped with a topology. There will come a time when we do equip vector spaces with topologies, and we will discuss series in that context. However, a linear combination will always mean a finite sum.

The next few results ought to be familiar, so we omit the proofs. Hoffman and Kunze [27] and Halmos [24] are good references for all this and more.

1.1.4 PROPOSITION *For a subset $\Lambda \subset \mathbf{E}$, the following are equivalent:*

(i) Λ is a Hamel basis for $\mathbf{E}$.

(ii) Λ is a maximal linearly independent set.

(iii) Λ is a minimal spanning set.

1.1.5 THEOREM (ZORN'S LEMMA) *Let $(\mathcal{F}, \leq)$ be a partially ordered set such that every totally ordered subset has an upper bound. Then $\mathcal{F}$ has a maximal element.*

> **Aside.** If $\mathcal{C}$ is a subset of $\mathcal{F}$, an upper bound for $\mathcal{C}$ is an element $x_0 \in \mathcal{F}$ (not necessarily in $\mathcal{C}$), with the property that $x \leq x_0$ for all $x \in \mathcal{C}$. A maximal element in $\mathcal{F}$ is an element m with the property that, if $x \in \mathcal{F}$ and $m \leq x$, then $m = x$. An important point that is often confusing is this: A maximal element of $\mathcal{F}$ need not be an upper bound for $\mathcal{F}$!

1.1.6 THEOREM *Every vector space has a Hamel basis. In fact, if $\Lambda_0 \subset \mathbf{E}$ is any linearly independent set, then there exists a Hamel basis Λ of $\mathbf{E}$ such that $\Lambda_0 \subset \Lambda$.*

1.1.7 EXAMPLE

(i) For $\mathbf{E} = \mathbb{K}^n$, we write $e_i := (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ (with 1 in the i^{th} position). The set $\{e_i : 1 \leq i \leq n\}$ is called the ***standard basis*** for $\mathbb{K}^n$.

(ii) Define

$$c_{00} := \{(x_n)_{n=1}^{\infty} : x_i \in \mathbb{K}, \text{ and there exists } N \in \mathbb{N} \text{ such that } x_i = 0 \text{ for all } i \geq N\}.$$

Members of c_{00} are sequences that are *eventually* zero (or equivalently, sequences with finite support). It is a vector space over $\mathbb{K}$ where the vector space operators are defined componentwise. Write e_i for the sequence

$$(e_i)_j = \delta_{i,j} = \begin{cases} 1 & : \text{ if } i = j, \\ 0 & : \text{ otherwise.} \end{cases}$$

Then $\{e_i : i \in \mathbb{N}\}$ is a basis for c_{00}.

(iii) Define

$$c_0 = \{(x_n)_{n=1}^{\infty} : x_i \in \mathbb{K}, \text{ and } \lim_{i \to \infty} x_i = 0\}.$$

Note that $\{e_i : i \in \mathbb{N}\}$ as above is a linearly independent set but *not* a basis for c_0. We will prove later that any basis of c_0 must be uncountable (see 5.1.4 Corollary). For now, though, give an example of an element in c_0 that cannot be expressed as a linear combination of the $\{e_i\}$.

(iv) Let $a, b \in \mathbb{R}$ with $a < b$, and define

$$C[a,b] := \{f : [a,b] \to \mathbb{K} \text{ continuous}\}.$$

This is a vector space over $\mathbb{K}$ under pointwise addition and scalar multiplication. For $n \geq 0$, let $e_n(x) := x^n$, then $\{e_n : n \geq 0\}$ is a linearly independent set, but it is not a basis for $C[a,b]$ (once again, do verify this).

More generally, if X is a compact, Hausdorff space, we set $C(X)$ to denote the space of continuous, $\mathbb{K}$-valued functions on X. This is a vector space under pointwise operations as well.

We will continue to develop this collection of examples as we go through the book. For the most part, all examples will fall into three 'types': finite dimensional vector spaces, sequence spaces and function spaces. While all of them may be profitably thought of as function spaces, it is more intuitive to think of them as different objects.

1.1.8 THEOREM *If* $\mathbf{E}$ *is a vector space, then any two Hamel bases of* $\mathbf{E}$ *have the same cardinality. This common number is called the* ***dimension*** *of* $\mathbf{E}$.

We omit the proof of this result. In the finite dimensional case, this is proved in Hoffman and Kunze [27, Section 2.3] while the proof in the infinite dimensional case is similar to that of Lemma 3.4.1 in Chapter 3.

1.1.9 DEFINITION Let $\mathbf{E}$ and $\mathbf{F}$ be two vector spaces.

(i) A function $T : \mathbf{E} \to \mathbf{F}$ is said to be a ***linear transformation*** or an ***operator*** if

$$T(\alpha x + y) = \alpha T(x) + T(y)$$

for all $x, y \in \mathbf{E}$ and $\alpha \in \mathbb{K}$.

(ii) We write $L(\mathbf{E}, \mathbf{F})$ for the set of all linear operators from $\mathbf{E}$ to $\mathbf{F}$. If $S, T \in L(\mathbf{E}, \mathbf{F})$ and $\alpha \in \mathbb{K}$, we define the operators $(S + T)$ and αS by

$$(S+T)(x) := S(x) + T(x), \text{ and } (\alpha S)(x) = \alpha S(x).$$

Clearly, this makes $L(\mathbf{E}, \mathbf{F})$ a $\mathbb{K}$-vector space.

(iii) If $\mathbf{F} = \mathbb{K}$, then a linear transformation $T : \mathbf{E} \to \mathbb{K}$ is called a ***linear functional***.

(iv) Given a linear transformation $T : \mathbf{E} \to \mathbf{F}$, there are two sets associated to T that we will often refer to:

$$\ker(T) := \{x \in \mathbf{E} : T(x) = 0\} \text{ and } \text{Range}(T) := \{T(x) : x \in \mathbf{E}\}.$$

It is easy to check that $\ker(T)$ and $\text{Range}(T)$ are subspaces of $\mathbf{E}$ and $\mathbf{F}$, respectively.

(v) A linear transformation $T : \mathbf{E} \to \mathbf{F}$ is said to be an ***isomorphism*** if T is bijective. If such a map exists, we write $\mathbf{E} \cong \mathbf{F}$.

1.1.10 Example

(i) Let $\mathbf{E} = \mathbb{K}^n, \mathbf{F} = \mathbb{K}^m$; then any $m \times n$ matrix A with entries in $\mathbb{K}$ defines a linear transformation $T_A : \mathbf{E} \to \mathbf{F}$ given by $x \mapsto A(x)$. Conversely, if $T \in L(\mathbf{E}, \mathbf{F})$, then the matrix whose columns are $\{T(e_i) : 1 \leq i \leq n\}$ defines an $m \times n$ matrix A such that $T = T_A$. If $M_{m\times n}(\mathbb{K})$ denotes the vector space of all such matrices, then there is an isomorphism of vector spaces

$$L(\mathbf{E}, \mathbf{F}) \cong M_{m\times n}(\mathbb{K})$$

given by $T_A \mapsto A$. If we replace the standard basis $\{e_1, e_2, \ldots, e_n\}$ by another basis Λ of $\mathbf{E}$, we get another isomorphism from $L(\mathbf{E}, \mathbf{F}) \to M_{m\times n}(\mathbb{K})$. Thus, the isomorphism is not canonical (it depends on the choice of basis).

(ii) Let $\mathbf{E} = c_{00}$ and define $\varphi : \mathbf{E} \to \mathbb{K}$ by

$$\varphi((x_j)) := \sum_{n=1}^{\infty} x_n.$$

Note that φ is well-defined and linear. Thus, $\varphi \in L(c_{00}, \mathbb{K})$.

(iii) Let $\mathbf{E} = C[a, b]$ and define $\varphi : \mathbf{E} \to \mathbb{K}$ by

$$\varphi(f) := \int_a^b f(t)dt.$$

Then $\varphi \in L(C[a, b], \mathbb{K})$.

(iv) Let $\mathbf{E} = \mathbf{F} = C[0, 1]$. Define $T : \mathbf{E} \to \mathbf{F}$ by

$$T(f)(x) := \int_0^x f(t)dt.$$

Note that T is well-defined (from Calculus) and linear. Thus $T \in L(\mathbf{E}, \mathbf{F})$.

1.1.11 DEFINITION Let $\mathbf{E}$ be a vector space and $\mathbf{F}$ be a subspace of $\mathbf{E}$.

(i) The ***quotient space***, denoted by $\mathbf{E}/\mathbf{F}$, is the quotient group, viewing $\mathbf{E}$ as an abelian group under addition and $\mathbf{F}$ as a (normal) subgroup. Note that $\mathbf{E}/\mathbf{F}$ has a natural vector space structure, with addition given by

$$(x+\mathbf{F})+(y+\mathbf{F}) := (x+y)+\mathbf{F},$$

and scalar multiplication given by $\alpha(x+\mathbf{F}) := \alpha x+\mathbf{F}$ for $\alpha \in \mathbb{K}$ and $x, y \in \mathbf{E}$.

(ii) The ***quotient map***, denoted by $\pi : \mathbf{E} \to \mathbf{E}/\mathbf{F}$, is given by $x \mapsto x+\mathbf{F}$. It is a surjective linear transformation such that $\ker(\pi) = \mathbf{F}$.

(iii) Furthermore, we define the ***codimension*** of $\mathbf{F}$ by $\operatorname{codim}(\mathbf{F}) := \dim(\mathbf{E}/\mathbf{F})$.

(iv) If $\operatorname{codim}(\mathbf{F}) = 1$, then we say that $\mathbf{F}$ is a ***hyperplane*** of $\mathbf{E}$.

Given a non-zero linear functional $\varphi : \mathbf{E} \to \mathbb{K}$, the subspace $\ker(\varphi)$ is a hyperplane in $\mathbf{E}$. Conversely, every hyperplane is of this form.

The next result is a simple consequence of Theorem 1.1.6, and we will omit its proof. Henceforth, we will write '$\mathbf{F} < \mathbf{E}$' to indicate that $\mathbf{F}$ is a subspace of $\mathbf{E}$.

1.1.12 PROPOSITION *Let* $\mathbf{E}$ *be a finite dimensional vector space and* $\mathbf{F} < \mathbf{E}$. *Then* $\operatorname{codim}(\mathbf{F}) = \dim(\mathbf{E}) - \dim(\mathbf{F})$.

One rarely mentions the First Isomorphism Theorem in the context of vector spaces but that is perhaps because the Rank–Nullity Theorem hogs the limelight. Also, the proof is completely analogous to the case of groups.

1.1.13 THEOREM (FIRST ISOMORPHISM THEOREM) *Let* $T : \mathbf{E} \to \mathbf{F}$ *be a linear transformation. Then*

(i) $\ker(T) < \mathbf{E}$ *and* $\operatorname{Range}(T) < \mathbf{F}$.

(ii) Furthermore, the map $\widehat{T} : \mathbf{E}/\ker(T) \to \operatorname{Range}(T)$ *given by*

$$x+\ker(T) \mapsto T(x)$$

is an isomorphism.

Let us now put the Rank–Nullity Theorem in its place. It is a direct consequence of the First Isomorphism Theorem, with a touch of Proposition 1.1.12.

1.1.14 THEOREM (RANK–NULLITY THEOREM) *If* $T : \mathbf{E} \to \mathbf{F}$ *is a linear transformation and* $\mathbf{E}$ *is finite dimensional, then* $\dim(\ker(T)) + \dim(\operatorname{Range}(T)) = \dim(\mathbf{E})$.

Recall that the ***nullity*** of T is $\dim(\ker(T))$ and the ***rank*** of T is $\dim(\text{Range}(T))$. Note that the dimension of the co-domain of the linear transformation plays no role in the Rank–Nullity Theorem; one merely needs the domain to be finite dimensional.

1.2 Review of Measure Theory

Historically, Lebesgue's theory of measure and integration provided great impetus to the then fledgling subject of Functional Analysis. In fact, it can be argued that Functional Analysis grew out of a need to understand the L^p spaces and operators between them. Therefore measure theory will be used liberally throughout this book.

However, to start with, we do not assume that the reader is necessarily familiar with all the nuances of measure theory. Instead, we would like to take a middle path. We will assume some familiarity with the notion of a measure, measurable functions, and basic integration theory, such as those available in the first few chapters of Royden [49] or Rudin [51]. As we go along, we will need more and more, and we hope that the reader will pick those things up as and when needed. For now, though, let us refresh our collective memories with the basic notions.

1.2.1 DEFINITION Let X be a set. A ***σ-algebra*** on X is a collection $\mathfrak{M}$ of subsets of X satisfying the following axioms:

(a) $\emptyset \in \mathfrak{M}$.

(b) If $E \in \mathfrak{M}$, then $E^c := X \setminus E \in \mathfrak{M}$.

(c) If $\{E_1, E_2, \ldots\}$ is a sequence of sets in $\mathfrak{M}$, then $\bigcup_{n=1}^{\infty} E_n \in \mathfrak{M}$.

The pair $(X, \mathfrak{M})$ is called a ***measurable space*** and the members of $\mathfrak{M}$ are called ***measurable sets***.

It is a useful fact (and one that is easy to prove) that if $\{\mathfrak{M}_\alpha : \alpha \in J\}$ is a family of σ-algebras on a set X, then the intersection $\bigcap_{\alpha \in J} \mathfrak{M}_\alpha$ is also a σ-algebra. In particular, if $\mathcal{S}$ is a collection of subsets of X, then there is a unique smallest σ-algebra on X that contains $\mathcal{S}$. This is called the σ-algebra ***generated by*** $\mathcal{S}$. The most important example of this phenomenon is the following.

1.2.2 DEFINITION Let X be a topological space. The σ-algebra generated by the topology on X is called the ***Borel σ-algebra*** on X and is denoted by $\mathfrak{B}_X$. The members of this σ-algebra are called ***Borel sets***.

Important examples of Borel sets are the following: A countable union of closed sets is called an F_σ-set and the countable intersection of open sets is called a G_δ-set.

1.2.3 DEFINITION Let $(X, \mathfrak{M})$ be a measurable space and Y be a topological space. A function $f : X \to Y$ is said to be ***measurable*** if $f^{-1}(U) \in \mathfrak{M}$ for every open set $U \subset Y$.

For the most part, measurable functions in this book will take values in $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$), where the latter is equipped with the usual topology. When it is important to make a distinction, we will refer to such functions as *real-measurable* or *complex-measurable*, as the case may be.

1.2.4 EXAMPLE

(i) Given a subset $E \subset X$, the ***characteristic function*** of E is the map $\chi_E : X \to \mathbb{R}$ given by

$$\chi_E(x) = \begin{cases} 1 & : \text{if } x \in E, \\ 0 & : \text{otherwise.} \end{cases}$$

Clearly, χ_E is a measurable function if and only if E is a measurable set.

(ii) More generally, a linear combination of characteristic functions of measurable sets is measurable. Such a function is called a ***simple function***. Alternatively, a simple function is a measurable function whose range is a finite set.

(iii) If X and Y are both topological spaces and we take $\mathfrak{M} = \mathfrak{B}_X$, then any measurable function $f : X \to Y$ is said to be ***Borel measurable***. Notice that every continuous function is Borel measurable (however, there are Borel measurable functions that are not continuous).

The class of measurable functions is closed under a number of operations, which we list below.

1.2.5 PROPOSITION *Let $(X, \mathfrak{M})$ be a measurable space.*

(i) If $f : X \to \mathbb{K}$, $g : X \to \mathbb{K}$ are measurable functions and $\alpha \in \mathbb{K}$, then $\alpha f + g$ is also measurable. So is the pointwise product $fg : X \to \mathbb{K}$, which is given by $x \mapsto f(x)g(x)$.

(ii) If $u : X \to \mathbb{R}$ and $v : X \to \mathbb{R}$ are real-measurable functions, then $f := u + iv$ is complex-measurable. Conversely, if $f : X \to \mathbb{C}$ is complex-measurable, then its real and imaginary parts are real-measurable functions.

(iii) *If* $f, g : X \to \mathbb{R}$ *are measurable, then so are* $\max\{f, g\}$ *and* $\min\{f, g\}$, *which are defined by* $\max\{f, g\}(x) := \max\{f(x), g(x)\}$, *and* $\min\{f, g\}(x) := \min\{f(x), g(x)\}$. *In particular,*

$$f^+ := \max\{f, 0\}, \text{ and } f^- := -\min\{f, 0\}$$

are both measurable.

(iv) *If* $f : X \to \mathbb{R}$ *is measurable, then so is* $|f| = f^+ + f^-$.

(v) *If* $\{f_n\}$ *are a sequence of* $\mathbb{K}$*-valued measurable functions, then* $\limsup_{n\to\infty} f_n$ *and* $\liminf_{n\to\infty} f_n$ *are both measurable. In particular, the pointwise limit of measurable functions (if it exists) is measurable.*

One important result that allows us to prove theorems about arbitrary measurable functions by first proving them for characteristic functions is the following.

1.2.6 THEOREM *Let* $f : X \to \mathbb{R}_+$ *be a non-negative measurable function. Then there is a sequence* (s_n) *of simple functions such that for each* $x \in X$, $(s_n(x))$ *is an increasing sequence of non-negative real numbers with* $\lim_{n\to\infty} s_n(x) = f(x)$.

Let us now turn to the notion of measure. This is a vast generalization of the idea of the 'volume' of a set and, unlike geometric notions of volume, it turns out to be flexible enough that we can prove interesting theorems about it.

1.2.7 DEFINITION Let $(X, \mathfrak{M})$ be a measurable space. A ***positive measure*** on $(X, \mathfrak{M})$ is a function $\mu : \mathfrak{M} \to [0, \infty]$ satisfying the following axioms.

(a) $\mu(\emptyset) = 0$.

(b) μ is *countably additive*: If $\{E_1, E_2, \ldots\}$ is a sequence of mutually disjoint sets in $\mathfrak{M}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

The triple $(X, \mathfrak{M}, \mu)$ is called a ***measure space***.

We will encounter both *real* and *complex* measures later on in the book, but the notion of a positive measure is the most basic. Therefore a positive measure will simply be referred to as a measure (without any qualification).

1.2.8 EXAMPLE

(i) Let X be any set and $x_0 \in X$ be a fixed point. Let $\mathfrak{M} := 2^X$ be the set of all subsets of X and let $\mu : \mathfrak{M} \to \mathbb{R}$ be the function

$$\mu(E) := \begin{cases} 1 & : \text{ if } x_0 \in E, \\ 0 & : \text{ if } x_0 \notin E. \end{cases}$$

This is called the ***Dirac measure*** at x_0 and is denoted by δ_{x_0}.

(ii) Let X be any set and $\mathfrak{M} := 2^X$ as above. Define $\mu : \mathfrak{M} \to [0, \infty]$ by

$$\mu(E) := \begin{cases} |E| & : \text{ if } E \text{ is finite,} \\ \infty & : \text{ otherwise.} \end{cases}$$

(where $|\cdot|$ denotes the cardinality function). It is clear that this is a measure on $(X, \mathfrak{M})$, and is called the ***counting measure***.

(iii) If X is a topological space, a measure on X is called a ***Borel measure*** if its domain contains $\mathfrak{B}_X$. Note that the domain of the measure may be larger than $\mathfrak{B}_X$ as well.

(iv) A measure μ on a measurable space $(X, \mathfrak{M})$ is said to be a ***finite measure*** if $\mu(X) < \infty$ and it is said to be ***σ-finite*** if X can be expressed as a countable union of sets of finite measure.

The most important measure is the ***Lebesgue measure*** on $\mathbb{R}$. No matter how you do it, the construction of the measure is long and complicated. However, we will describe it in enough detail so as to have a working understanding of it.

Consider $\mathbb{R}$, equipped with the usual topology. Then there is a σ-algebra $\mathfrak{L}$ which contains $\mathfrak{B}_{\mathbb{R}}$, and a positive measure $m : \mathfrak{L} \to [0, \infty]$ satisfying the following properties:

(a) If $a, b \in \mathbb{R}$ with $a < b$, then $m([a, b)) = (b - a)$.

(b) If $E \in \mathfrak{L}$ and $x \in \mathbb{R}$, then $E + x \in \mathfrak{L}$ and $m(E + x) = m(E)$. This property is called ***translation invariance*** of the measure m (here, $E + x$ is the set $\{y + x : y \in E\}$).

(c) If $E \in \mathfrak{L}$, then

$$m(E) = \inf\{m(U) : U \text{ open}, E \subset U\} = \sup\{m(K) : K \text{ compact}, K \subset E\}.$$

This property is called ***regularity*** of the measure m.