## Contents

**Preface** xi  
**Introduction** 1  
**Preliminaries** 17  

### Chapter 1. Coxeter groups and reflection arrangements 19  
1.1. Face-types, flat-types, biface-types 20  
1.2. Lune-types and bilune-types 29  
1.3. Stabilizer subgroups of a Coxeter group 34  
1.4. Gallery distance, gate property, length 35  
1.5. Modules over the Coxeter group 37  
1.6. Descents, global descents, shuffles 38  
1.7. Groupoid of biface-types 40  
1.8. Face-type enumeration 48  
1.9. Enumeration in the Coxeter group algebra 60  
1.10. Invariant flat-, lune-, bilune-incidence algebras 63  
1.11. Noncommutative Zaslavsky formulas and $q$-analogues 75  
1.12. Invariant Birkhoff, Tits, Janus algebras 77  
1.13. Invariant Lie and Zie elements 84  
1.15. Coordinate arrangement 87  
1.16. Braid arrangement 90  
1.17. Notes 95  

### Part I. Coxeter species 99  

### Chapter 2. Coxeter species and Coxeter bimonoids 101  
2.1. Support and type morphisms 104  
2.2. Induced and local morphisms 109  
2.3. Coxeter species 110  
2.4. Coxeter monoids and Coxeter comonoids 116  
2.5. Coxeter bimonoids 122  
2.6. (Co)commutative Coxeter (co)monoids 127  
2.7. Species vs Coxeter species 135  
2.8. Signature functors on Coxeter species 137  
2.9. Coxeter (co, bi)monoids as functor categories 139
2.10. Bimonads on Coxeter species 147
2.11. Opposite transformation 153
2.12. Joyal species and Coxeter species 154
2.13. Notes 156

Chapter 3. Basic theory of Coxeter bimonoids 159
3.1. Cauchy powers of a Coxeter species 159
3.2. Primitive and decomposable filtrations 160
3.3. Universal constructions of Coxeter bimonoids 164
3.4. (Co)abelianizations of Coxeter (co)monoids 174
3.5. Hadamard product. Convolution Coxeter monoid 176
3.6. Internal hom for Coxeter comonoids 180
3.7. Universal measuring Coxeter comonoid 183
3.8. Exp-log between (co)derivations and (co)monoid morphisms 185
3.9. Logarithm of the identity map 190
3.10. Exp-log between primitive and group-like series 193
3.11. Norm transformation on Coxeter bimonoids 200
3.12. Characteristic operations by faces 203
3.13. Algebra modules and Coxeter bimonoids 205
3.14. Antipodes of Coxeter bimonoids 210
3.15. Notes 213

Chapter 4. Examples of Coxeter bimonoids 216
4.1. Coxeter species characteristic of chambers 216
4.2. Exponential Coxeter bimonoid 218
4.3. Coxeter bimonoid of chambers 222
4.4. Coxeter bimonoid of flats 225
4.5. Coxeter bimonoid of faces 228
4.6. Coxeter bimonoid of top-nested faces 233
4.7. Coxeter bimonoid of pairs of chambers 235
4.8. Coxeter Lie species and Coxeter Zie species 237
4.9. Decoration functor on Coxeter species 238
4.10. Notes 244

Chapter 5. Coxeter operads 246
5.1. Coxeter dispecies 247
5.2. Coxeter operads 248
5.3. Hadamard product 252
5.4. Coxeter operad monoids 252
5.5. Coxeter commutative, associative, Lie operads 255
5.6. Notes 260

Chapter 6. Coxeter Lie monoids 261
6.1. Coxeter Lie monoids and Coxeter Lie comonoids 262
6.2. (Co)commutator (co)bracket 264
6.3. Primitive and indecomposable parts of Coxeter bimonoids 265
6.4. (Co)free Coxeter Lie (co)monoids 266
CONTENTS

6.5. Universal (co)enveloping Coxeter (co)monoids 269
6.6. Joyal Lie monoids and Coxeter Lie monoids 271
6.7. Notes 272

Chapter 7. Structure theory of Coxeter bimonoids 273
7.1. Borel–Hopf for (co)commutative Coxeter bimonoids 274
7.2. Leray–Samelson for bicommutative Coxeter bimonoids 275
7.3. Rigidity of Coxeter q-bimonoids for q not a root of unity 277
7.4. Loday–Ronco for Coxeter 0-bimonoids 279
7.5. Hoffman–Newman–Radford 280
7.6. Poincaré–Birkhoff–Witt 284
7.7. Cartier–Mihno–Moore 287
7.8. Coxeter bimonoids for a rank-one arrangement 288
7.9. Notes 297

Part II. Coxeter spaces 299

Chapter 8. Coxeter spaces and Coxeter bialgebras 301
8.1. Support and type morphisms 304
8.2. Induced and local morphisms 311
8.3. Opposite and adjacent support-type morphisms 314
8.4. Coxeter spaces 317
8.5. Coxeter algebras and Coxeter coalgebras 324
8.6. Coxeter bialgebras 331
8.7. Coxeter q-bialgebras 341
8.8. (Co)commutative Coxeter (co)algebras 347
8.9. Coxeter bialgebras and face-type enumeration 355
8.10. Signature functor on Coxeter spaces 356
8.11. Coxeter (co, bi)algebras as functor categories 358
8.12. Bimonads on Coxeter spaces 365
8.13. Opposite transformation 372
8.15. Notes 379

Chapter 9. Basic theory of Coxeter bialgebras 382
9.1. Cauchy powers of a Coxeter space 383
9.2. Primitive and decomposable filtrations 385
9.3. Universal constructions of Coxeter bialgebras 388
9.4. (Co)abelianizations of Coxeter (co)algebras 399
9.5. Hadamard product. Convolution Coxeter algebra 403
9.6. Internal hom for Coxeter coalgebras 406
9.7. Universal measuring Coxeter coalgebra 409
9.8. Exp-log between (co)derivations and (co)algebra morphisms 412
9.9. Logarithm of the identity map 418
9.10. Exp-log between primitive and group-like series 421
9.11. Norm transformation on Coxeter bialgebras 430
9.12. Bialgebra axiom on the primitive part 437
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.13</td>
<td>Characteristic operations by face-types</td>
<td>439</td>
</tr>
<tr>
<td>9.14</td>
<td>Antipodes of Coxeter bialgebras</td>
<td>445</td>
</tr>
<tr>
<td>9.15</td>
<td>Notes</td>
<td>451</td>
</tr>
<tr>
<td>10.1</td>
<td>Examples of Coxeter bialgebras</td>
<td>456</td>
</tr>
<tr>
<td>10.2</td>
<td>Coxeter space characteristic of chambers</td>
<td>457</td>
</tr>
<tr>
<td>10.3</td>
<td>Coxeter bialgebra of polynomials</td>
<td>459</td>
</tr>
<tr>
<td>10.4</td>
<td>Tensor, shuffle, symmetric, exterior Coxeter bialgebras</td>
<td>471</td>
</tr>
<tr>
<td>10.5</td>
<td>Coxeter bialgebra of face-types</td>
<td>483</td>
</tr>
<tr>
<td>10.6</td>
<td>Coxeter bialgebra of flat-types</td>
<td>505</td>
</tr>
<tr>
<td>10.7</td>
<td>Coxeter bialgebra of symmetries</td>
<td>510</td>
</tr>
<tr>
<td>10.8</td>
<td>Notes</td>
<td>515</td>
</tr>
<tr>
<td>11.1</td>
<td>Coxeter operad algebras</td>
<td>522</td>
</tr>
<tr>
<td>11.2</td>
<td>Nested flats, nested top-lunes, nested face-types</td>
<td>523</td>
</tr>
<tr>
<td>11.3</td>
<td>Coxeter dispecies</td>
<td>529</td>
</tr>
<tr>
<td>11.4</td>
<td>Coxeter operads</td>
<td>531</td>
</tr>
<tr>
<td>11.5</td>
<td>Coxeter spaces as a module over Coxeter dispecies</td>
<td>537</td>
</tr>
<tr>
<td>11.6</td>
<td>Coxeter operad algebras</td>
<td>542</td>
</tr>
<tr>
<td>11.7</td>
<td>Coxeter commutative, associative, Lie operads revisited</td>
<td>547</td>
</tr>
<tr>
<td>11.8</td>
<td>Signature functor on Coxeter dispecies</td>
<td>554</td>
</tr>
<tr>
<td>11.9</td>
<td>May operads and Coxeter operads</td>
<td>557</td>
</tr>
<tr>
<td>11.10</td>
<td>Notes</td>
<td>560</td>
</tr>
<tr>
<td>12.1</td>
<td>Coxeter Lie algebras</td>
<td>561</td>
</tr>
<tr>
<td>12.2</td>
<td>(Co)commutator (co)bracket</td>
<td>567</td>
</tr>
<tr>
<td>12.3</td>
<td>Primitive and indecomposable parts of Coxeter bialgebras</td>
<td>568</td>
</tr>
<tr>
<td>12.4</td>
<td>(Co)free Coxeter Lie (co)algebras</td>
<td>571</td>
</tr>
<tr>
<td>12.5</td>
<td>Universal (co)enveloping Coxeter (co)algebras</td>
<td>576</td>
</tr>
<tr>
<td>12.6</td>
<td>Graded Lie algebras and Coxeter Lie algebras</td>
<td>582</td>
</tr>
<tr>
<td>12.7</td>
<td>Notes</td>
<td>585</td>
</tr>
<tr>
<td>13.1</td>
<td>Structure theory of Coxeter bialgebras</td>
<td>586</td>
</tr>
<tr>
<td>13.2</td>
<td>Borel–Hopf for (co)commutative Coxeter bialgebras</td>
<td>588</td>
</tr>
<tr>
<td>13.3</td>
<td>Leray–Samelson for bicommutative Coxeter bialgebras</td>
<td>595</td>
</tr>
<tr>
<td>13.4</td>
<td>Rigidity of Coxeter $q$-bialgebras for $q$ not a root of unity</td>
<td>600</td>
</tr>
<tr>
<td>13.5</td>
<td>Loday–Ronco for Coxeter 0-bialgebras</td>
<td>606</td>
</tr>
<tr>
<td>13.6</td>
<td>Hoffman–Newman–Radford</td>
<td>607</td>
</tr>
<tr>
<td>13.7</td>
<td>Poincaré–Birkhoff–Witt</td>
<td>614</td>
</tr>
<tr>
<td>13.8</td>
<td>Cartier–Milnor–Moore</td>
<td>620</td>
</tr>
<tr>
<td>13.9</td>
<td>Coxeter bialgebras for a rank-one arrangement</td>
<td>623</td>
</tr>
<tr>
<td>13.10</td>
<td>Notes</td>
<td>638</td>
</tr>
<tr>
<td>Part III.</td>
<td>Fock functors</td>
<td>641</td>
</tr>
<tr>
<td>14.1</td>
<td>Full Fock functors</td>
<td>644</td>
</tr>
</tbody>
</table>
14.2. Deformed full Fock functors 651
14.3. Full Fock functors and signature functors 653
14.4. Bosonic and fermionic Fock functors 656
14.5. Full Fock functors from bosonic Fock functors 661
14.6. Fock functors and commutativity 662
14.7. Norm transformation between Fock functors 671
14.8. Anyonic Fock functor 674
14.9. Fock functors and Coxeter operads 676
14.10. Decorated Fock functors 682
14.11. Notes 684

Chapter 15. Coxeter bimonoids and Coxeter bialgebras 687
15.1. Coxeter bimonoids to Coxeter bialgebras 689
15.2. Commutativity under Fock functors 694
15.3. Norm map between Fock spaces 695
15.4. (Co)freeness under Fock functors 697
15.5. Nonnegativity of the Boolean transform 701
15.6. Coxeter operad monoids to Coxeter operad algebras 702
15.7. Coxeter Lie monoids to Coxeter Lie algebras 705
15.8. Primitive and indecomposable parts under Fock functors 708
15.9. Universal enveloping constructions under Fock functors 710
15.10. Notes 712

Chapter 16. Adjoints of Fock functors 714
16.1. Trivialization functor 717
16.2. Hadamard product with chambers 721
16.3. Adjoints of bosonic Fock functors 722
16.4. Lifts of adjunctions involving bosonic Fock functors 725
16.5. Bosonic Fock functors and abelianization 739
16.6. Adjoints preserving commutativity 740
16.7. Adjoints of fermionic Fock functors 745
16.8. Adjoints of full Fock functors 745
16.9. Coxeter operad algebras to Coxeter operad monoids 750
16.10. Coxeter Lie algebras to Coxeter Lie monoids 753
16.11. (Co)freeness under Fock functor adjoints 756
16.12. Notes 757

Chapter 17. Structure theory under Fock functors 759
17.1. Series of Coxeter species and Coxeter spaces 759
17.2. Actions of the invariant bilune-incidence algebra 762
17.3. Internal hom for Coxeter species and Coxeter spaces 766
17.4. Norm maps on Coxeter species and Coxeter spaces 769
17.5. Transformation between bosonic Fock functor adjoints 772
17.6. Antipodes under Fock functors 776
17.7. Structure theorems under Fock functors 778
17.8. Lie theory under Fock functors 782
17.9. Notes 785
Chapter 18. Examples of Fock spaces

18.1. Coxeter space characteristic of chambers as a Fock space 787
18.2. Coxeter bialgebra of polynomials as a Fock space 788
18.3. Tensor Coxeter bialgebra as a Fock space 794
18.4. Coxeter bialgebra of face-types as a bosonic Fock space 798
18.5. Coxeter bialgebra of flat-types as a bosonic Fock space 804
18.6. Coxeter bialgebra of symmetries as a bosonic Fock space 806
18.7. Fock functors for a rank-one arrangement 807
18.8. Notes 815

Appendix A. Category theory

A.1. Bimonads and bilax functors 816
A.2. Semidirect products 821
A.3. Notes 824

References 826

List of Notations 842
List of Tables 852
List of Figures 855
List of Summaries 857
Author Index 858
Subject Index 863