COXETER BIALGEBRAS

The goal of this monograph is to develop Hopf theory in the setting of a real reflection arrangement. The central notion is that of a Coxeter bialgebra which generalizes the classical notion of a connected graded Hopf algebra. The authors also introduce the more structured notion of a Coxeter bimonoid and connect the two notions via a family of functors called Fock functors. These generalize similar functors connecting Hopf monoids in the category of Joyal species and connected graded Hopf algebras.

The building blocks of the theory are geometric objects associated to a reflection arrangement such as faces, flats, lunes, and their orbits under the action of the Coxeter group. A generalized notion of zeta and Möbius function play a fundamental role in all aspects of the theory, including exp-log correspondences and results such as the Poincaré–Birkhoff–Witt theorem. The Tits algebra and its invariant subalgebra also play key roles.

This monograph opens a new chapter in Coxeter theory as well as in Hopf theory, connecting the two. It also relates fruitfully to many other areas of mathematics such as discrete geometry, semigroup theory, associative algebras, algebraic Lie theory, operads, and category theory. It is carefully written, with effective use of tables, diagrams, pictures, and summaries. It will be of interest to students and researchers alike.

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Coxeter Bialgebras

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Preface

Goal

Species and bimonoids for linear hyperplane arrangements were introduced and studied in our previous monograph [**11**]. It marks the beginning of Hopf and Lie theory for arrangements. Faces of the arrangement play a critical role in this theory. The Tits product on faces, for instance, enters into the formulation of the bimonoid axiom.

The present work aims to introduce and study Coxeter species and Coxeter spaces. These are defined in the special setting of linear reflection arrangements. We formulate Coxeter species using faces; they carry more structure than species, the additional structure comes from the action of the Coxeter group of the reflection arrangement on faces. We formulate Coxeter spaces using face-types; they carry less structure than Coxeter species, making their study considerably harder. Similarly, we define and study Coxeter bimonoids and Coxeter bialgebras.

The classical picture is of Joyal species and graded vector spaces, and of Joyal bimonoids and graded bialgebras. This picture connects to the one above via braid arrangements (which are a family of linear reflection arrangements). A summary is provided in Table I below.

Table I. Coxeter bimonoids and Joyal bimonoids.

Starting data	Objects of interest	
Hyperplane arrangement	Species Bimonoids	
Reflection arrangement	Coxeter species Coxeter bimonoids	Coxeter spaces Coxeter bialgebras
Braid arrangement (classical)	Joyal species Joyal bimonoids	Graded vector spaces Graded bialgebras

The relevance of braid arrangements to Hopf theory was brought out and promoted in all our earlier works [**6**], [**7**], [**8**]. Along with combinatorial notions such as shuffles and integer compositions, we emphasized related geometric notions such as the Tits product, gate property, distance functions, lunes, and so on, thus making connections to Coxeter theory more transparent, and paving the way for [**11**] and the present work.

The theory of operads for linear hyperplane arrangements was also initiated in our earlier monographs [**10**, Section 15.9], [**11**, Chapter 4]. In the present work, we introduce the notion of a Coxeter operad for a linear reflection arrangement. Coxeter operads relate to classical operads (which we call May operads) via braid arrangements. See the summary in Table II below.

Table II. Coxeter operads and May operads.

Starting data	Objects of interest
Hyperplane arrangement	Operads
Reflection arrangement	Coxeter operads
Braid arrangement (classical)	May operads

Historical note

The notion of a Coxeter bialgebra matured out of [**6**], where Coxeter theory and Hopf theory were linked together for the first time. In a similar vein, the notion of a Coxeter bimonoid matured out of [**7**], where the theory of Joyal bimonoids was laid out in detail. We formulated Coxeter bimonoids, Coxeter bialgebras and Fock functors linking them in the year 2011, and their basic theory became clear to us by the time [**8**] was published in 2013. It necessarily involved Coxeter Lie monoids and Coxeter Lie algebras which can be viewed as objects of algebraic Lie theory. There was also the intertwining story of Coxeter operads.

Writing all this up in a coherent manner presented us with many expository problems. We first needed to write [**10**] which could then serve as a general reference for hyperplane arrangements. This book was published in 2017 after a lot of administrative delays. Next, we felt that it would be easier to explain the theory of species, operads, Lie monoids and bimonoids for hyperplane arrangements which was less structured but worked in a more general setting and had direct connections to the abstract theory of semigroups. This led to the monograph [**11**] published in 2020. While working on this monograph, we also carefully looked at the early history of Hopf algebras; interested readers can see the end-of-chapter notes in [**11**]. Our nomenclature for structure theorems in Hopf theory is based on this study.

The above developments were also instrumental in our discovery of the category of lunes, of noncommutative zeta and Möbius functions, of the q-Janus algebra, and of the groupoid of biface-types.

Synopsis

We now briefly summarize the contents of this monograph.

Objects. For a linear reflection arrangement, we introduce the category of Coxeter species and notions of Coxeter monoid, Coxeter comonoid, Coxeter bimonoid, Coxeter Lie monoid therein. For a scalar q, we introduce Coxeter q-bimonoids, with $q = 1$ recovering Coxeter bimonoids. For $q = -1$, we use the term signed Coxeter bimonoid. These generalize corresponding notions in Joyal species [**7**, Chapters 8 and 9] from braid arrangements to any linear reflection arrangement. Similarly, we introduce the category of Coxeter spaces and notions of Coxeter algebra, Coxeter coalgebra, (signed) Coxeter bialgebra, Coxeter q-bialgebra, Coxeter Lie algebra therein. These generalize familiar notions in graded vector spaces, namely, graded algebra, graded coalgebra, (signed) graded bialgebra, graded q-bialgebra, graded Lie algebra.

For a linear reflection arrangement, we introduce the notion of a Coxeter operad generalizing the classical notion of a May operad pertaining to braid arrangements. We define the Coxeter commutative operad, Coxeter associative operad, Coxeter Lie operad. Modules over these Coxeter operads in the category of Coxeter species are commutative Coxeter monoids, Coxeter monoids, Coxeter Lie monoids, respectively, and in the category of Coxeter spaces are commutative Coxeter algebras, Coxeter algebras, Coxeter Lie algebras, respectively. We also consider their dual Coxeter cooperads whose comodules yield Coxeter comonoids, and so on. Coxeter (co)operads give rise to (co)monads on Coxeter species and on Coxeter spaces; thus, Coxeter monoids and Coxeter algebras can also be viewed as algebras over a monad, and so on.

Theory. We formulate Coxeter bimonoids in terms of faces and Coxeter bialgebras in terms of face-types; their theories proceed largely in parallel with each other. (There are alternative formulations of Coxeter bimonoids in terms of flats and lunes, and of Coxeter bialgebras in terms of top-nested faces, top-lunes.) We discuss primitive and decomposable filtrations, universal constructions, (co)abelianization constructions, Hadamard product, enrichment over Coxeter comonoids and Coxeter coalgebras, the antipode and Takeuchi formula, exp-log correspondences, q-norm map with connections to the q-logarithm, characteristic operations. We present structure results such as the Loday–Ronco theorem, its q-generalization for q not a root of unity, the Leray–Samelson theorem, the Borel–Hopf theorem, the Hoffman–Newman–Radford rigidity theorems, and the Poincaré–Birkhoff–Witt and Cartier–Milnor–Moore theorems. Noncommutative zeta and Möbius functions, their q-analogues and representation theory of the (invariant) Tits algebra and (invariant) q-Janus algebra play a prominent role.

Examples. Examples of Coxeter bimonoids include the exponential Coxeter bimonoid and Coxeter bimonoids of chambers, flats, faces, top-nested faces, top-lunes and pairs of chambers. These generalize the exponential Joyal bimonoid and Joyal bimonoids of linear orders, set partitions, set compositions, linear set compositions, linear set partitions, and pairs of linear orders [**7**, Chapter 12].

Examples of Coxeter bialgebras include the Coxeter bialgebra of polynomials, the tensor and shuffle Coxeter bialgebras and Coxeter bialgebras of flat-types, face-types, and symmetries. These generalize the Hopf algebra of polynomials, the tensor and shuffle Hopf algebras, and Hopf algebras of symmetric functions, noncommutative symmetric functions, quasisymmetric functions and permutations. Interestingly, several identities related to face-type enumeration follow from the study of these Coxeter bialgebras.

Interconnections. The category of Coxeter species is richer than that of Coxeter spaces (just as the category of Joyal species is in relation to that of graded vector spaces). Similarities and differences between the two categories can be conceptually understood via a family of functors called Fock functors which can be pictured as

$$\{\text{Coxeter species}\} \xrightarrow{\text{Fock functors}} \{\text{Coxeter spaces}\}.$$

Fock functors carry Coxeter bimonoids to Coxeter bialgebras. For instance, they carry the exponential Coxeter bimonoid to the Coxeter bialgebra of polynomials. Adjoints of Fock functors go in the other direction starting with

$$\{\text{Coxeter spaces}\} \xrightarrow{\text{Fock functor adjoints}} \{\text{Coxeter species}\}.$$

Such adjoints allow us to exhibit Coxeter bialgebras as a reflective and coreflective subcategory of Coxeter bimonoids. They provide a theoretical tool to establish results on Coxeter bialgebras using corresponding results on Coxeter bimonoids. We provide many illustrations of this nature.

Prerequisites

We assume basic familiarity with posets, monoids, groups, group representations and (co)invariants, associative algebras (such as monoid algebras, group algebras, incidence algebras), modules over algebras. Other than that, prerequisites for reading this book pertain to three main areas: category theory, Hopf and Lie theory, reflection arrangements and Coxeter groups. They are elaborated below.

Category theory. We assume familiarity with category theory at the level of [**25**], [**226**], [**307**], [**309**]. Concepts such as functors, natural transformations, groupoids, functor categories, equivalences between categories, presentation of categories, adjunctions and universal properties are used without explanation. Commutative diagrams form an essential component of our exposition. Bimonads and bilax functors, and semidirect products are briefly recalled in Appendix A. Some other concepts which appear in this book are $\Bbbk$-linear categories, enrichment over monoidal categories, double categories.

Hopf and Lie theory. This book presents Hopf and Lie theory first for Coxeter species and then for Coxeter spaces. The theory for Coxeter species (for reflection arrangements) proceeds in analogy with the theory for species (for hyperplane arrangements) developed in detail in [**11**]. To appreciate the role of the Tits monoid and related geometric ideas in everything that we do, some basic familiarity with [**11**] is highly desirable. Having said this, our discussion on Coxeter species, though compact, is fairly self-contained, with occasional references to [**11**].

Coxeter bimonoids generalize Joyal bimonoids [**7**], [**8**], while Coxeter bialgebras generalize connected graded Hopf algebras [**345**]. Since we develop everything from first principles, familiarity with either Joyal bimonoids or Hopf algebras is not strictly required, though some exposure to these notions is useful for motivational purposes.

Reflection arrangements and Coxeter groups. We assume familiarity with reflection arrangements at the level of [**10**, Chapters 1, 5, 6]. Some specialized topics that we require include noncommutative zeta and Möbius functions, representation theory of the (invariant) Tits algebra, and identities related to face-type enumeration [**10**, Chapters 9, 15, 16]. To keep the book self-contained, this material (along with some new results) is reviewed here in an introductory chapter. The introductory chapter of [**11**] on hyperplane arrangements is also a valuable reference.

Readership

This monograph would be of interest to students and researchers working in the areas of hyperplane arrangements, Coxeter groups, discrete geometry, semigroup theory, associative algebras, Joyal species, May operads, Hopf algebras, algebraic Lie theory, category theory.

Organization

The book is organized into three parts:

- Part I introduces Coxeter species and Coxeter bimonoids and develops their basic structure theory,
- Part II does the same for Coxeter spaces and Coxeter bialgebras,
- Part III develops the theory of Fock functors which connects Coxeter species and Coxeter spaces.

Parts I and II are independent of each other. Objects in Part I are more structured than objects in Part II making them easier to study. Nonetheless, in terms of presentation, we treat both of them equally. Part III connects the two. The organization of Part I is similar to that of [**11**], and does for Coxeter species what [**11**] did for species. The two expositions supplement each other and enable the reader to benefit from both. Parts I and III may also be compared to [**7**, Parts II and III], and do for Coxeter species and Coxeter spaces what [**7**] did for Joyal species and graded vector spaces.

The chart of interdependence of chapters is shown in Figure I. Chapter 1 is introductory in nature. The dotted horizontal lines bring out the parallel between Parts I and II. Part III is to be largely read in order. The thick horizontal arrow indicates that it depends on both Parts I and II. Chapters 4, 10, 18 are not shown in the chart; they consist of examples and can be read in parallel with the theory developed in other chapters.

Tables, pictures, summaries form an essential component of our exposition. Examples and exercises (of varying levels of difficulty and with generous hints) are interspersed throughout the book. A Notes section at the end of

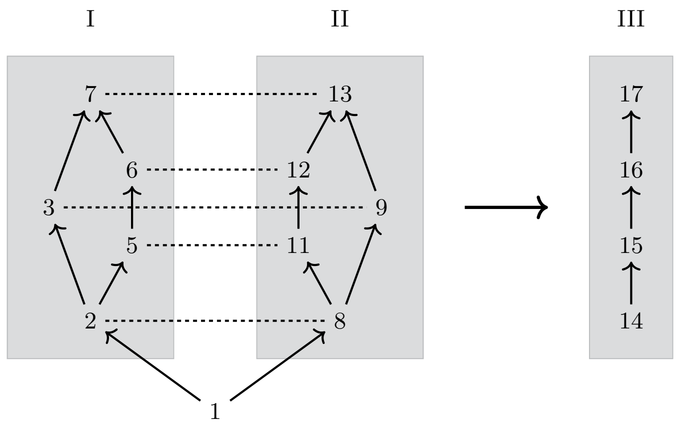

Figure I. Chart of interdependence of chapters.

each chapter connects to the classical literature with explanations and references. A list of notations, a list of tables, a list of figures, a list of summaries, an author index, a subject index are provided at the end of the book.

Teaching

The book is suitable for a two-semester sequence at the graduate level. It can also be used selectively for courses of shorter duration or theme-based seminars. Here are some guidelines; they can also be used for self-study.

For a systematic approach:

- The introductory Chapter 1, supplemented by [**10**], can be used to teach a course on Coxeter groups. It contains many new ideas and perspectives not to be found in standard books on Coxeter theory.
- Alternatively, one can focus entirely on Part I on Coxeter species and supplement it with material on species in [**11**]. Numerous subplans are possible here; see in particular those outlined on [**11**, Preface, page xvi]. We elaborate slightly. First develop basic concepts from Chapter 2 in parallel with some examples from Chapter 4. Then decide on which topics to cover from Chapter 3 and move ahead from there.
- More ambitiously, one can focus entirely on Part II on Coxeter spaces, and follow the same plan as above. Namely: First develop basic concepts from Chapter 8 in parallel with some examples from Chapter 10. Then decide on which topics to cover from Chapter 9 and move ahead.
- Alternatively, one can focus on Part III on Fock functors. The way to go would be to first develop basic familiarity with the main objects in the first two parts, namely, Coxeter bimonoids and Coxeter bialgebras, and then study how they interact with each other via Fock functors.
- Chapters 5 and 11, supplemented by [**11**, Chapter 4] can be used to teach a course on Coxeter operads.

For an exploratory approach:

- One can focus on a specific aspect of Hopf theory such as the Hadamard product or exp-log correspondences for both Coxeter species and Coxeter spaces and see how they compare.
- A course can be built around a rich example such as the Coxeter bialgebra of face-types, using it to grasp different aspects of the theory.
- One can also focus on a specific topic from Chapter 1 and explore how it connects to Coxeter bialgebras. Possible topics are shuffles, modules over the Coxeter group, face-type enumeration, zeta and Möbius functions, invariant Tits and invariant q-Janus algebras.
- One can approach everything through the lens of category theory. For instance, one can focus on bimonads and bilax functors, and see how the main objects in the book fit into this framework. A possible subtopic here could be adjunctions and universal constructions. Similarly, one can study the different finite categories (groupoids) built out of geometric objects such as faces, flats, face-types, and so on.
- One can also look at "visual" aspects of Coxeter bialgebras. We explain many concepts, calculations, arguments with pictures (in the groupoid of biface-types) involving vertices, edges, triangles, and arrows. The reader can come up with alternative pictures by changing parameter values, and also visualize one dimension higher with tetrahedra.

The book can also be used as an additional reference while teaching standard courses on category theory, Hopf algebras, Lie algebras, Joyal species, and May operads.

New developments

The extension of Hopf theory to hyperplane arrangements done in [**11**] brought forth many new ideas; see [**11**, Preface, pages xvii–xix]. The special setting of reflection arrangements pursued in the present book substantiates these ideas as well as adds several new features. They are listed below.

Reflection arrangements.

- emergence of the groupoid of biface-types with face-types as objects and biface-types as morphisms,
- role of the invariant Tits and invariant q-Janus algebras and connection of their structure constants to face-type enumeration,
- emergence of identities in the group algebra of the Coxeter group related to shuffles.

Coxeter species and Coxeter spaces.

- distinction between species for a hyperplane arrangement, and Coxeter species for a reflection arrangement,
- emphasis on "higher operations" involving faces and face-types as opposed to "binary operations" involving vertices and vertex-types,
- set-theoretic nature of the bimonoid axiom for Coxeter bimonoids as opposed to linear nature of the bialgebra axiom for Coxeter bialgebras;

moreover, the former involves only support morphisms, while the latter mixes support and type morphisms,
- connection between the isomorphism problem for the general q-norm map on Coxeter q-bimonoids (or on Coxeter q-bialgebras) and the q-logarithm for q not a root of unity,
- connection between representation theory of algebras and Hopf theory that relates Coxeter–Birkhoff algebra modules to bicommutative Coxeter bimonoids, Coxeter–Tits algebra modules to (co)commutative Coxeter bimonoids, Coxeter–Janus algebra modules to Coxeter bimonoids,

- construction of the decoration functor on Coxeter species starting with a module over the Coxeter group,
- embedding the category of Coxeter group modules inside the categories of Coxeter species and Coxeter spaces,

- clarity on how field characteristic issues enter into the Coxeter story via invariant noncommutative zeta and Möbius functions,
- deducing results for Coxeter species from those for species, thus, avoiding field characteristic issues wherever possible,

- distinction between the o-signature functor and u-signature functor on Coxeter species constructed from signature spaces over and under flats,
- seamless passage between unsigned and signed worlds of both Coxeter species and Coxeter spaces via the u-signature functor,

- derivation of several enumeration identities in Coxeter theory involving face-types and shuffles from existence of specific Coxeter bialgebras.

Coxeter operads.

- distinction between operads for a hyperplane arrangement, and Coxeter operads for a reflection arrangement,
- emergence of the monoidal category of Coxeter dispecies under the substitution product leading to the notion of Coxeter operads,
- interpreting Coxeter species and Coxeter spaces as module categories over Coxeter dispecies leading to notions of Coxeter operad monoids and Coxeter operad algebras,
- introduction of the Coxeter commutative, associative, Lie operads involving a rich interplay of flats, top-lunes, face-types (which is specific to the Coxeter context).

Fock functors.

- $\Bbbk$-linearity of the categories of Coxeter (co, bi)monoids and Coxeter (co, bi)algebras, and also of Fock functors and their adjoints (made possible by the absence of the traditional tensor product of vector spaces),
- interactions of the bosonic Fock functor with (primitive and group-like) series of Coxeter bimonoids and of Coxeter bialgebras and their exp-log correspondences,

- interactions of Fock functors with the general q-norm map on Coxeter q-bimonoids and on Coxeter q-bialgebras,
- viewing Coxeter bialgebras as a reflective and coreflective subcategory of Coxeter bimonoids via adjoints of bosonic Fock functors, and deducing results for Coxeter bialgebras from those for Coxeter bimonoids,
- proving nonnegativity of the Boolean transform of the Hilbert type series of Coxeter bimonoids using Fock functors.

Perspectives and future work

This monograph along with the previous one [**11**] opens a vast new area of research. Needless to say, Coxeter bimonoids, Coxeter bialgebras, Fock functors, Coxeter operads are fundamental objects worthy of further study. Two important perspectives on this work are summarized below.

(1) It gives geometric insights into the theory of connected graded Hopf algebras, with combinatorics of integer compositions interpreted as geometry of face-types. This upgrades definitions (for instance, by employing higher operations), improves known results (for instance, by systematically employing zeta and Möbius series), and points to new results (for instance, on connected graded q-bialgebras). These are indicated in the end-of-chapter notes and we plan to explain this in detail in a separate work. Similar comments apply to the theory of May operads.

(2) It brings a Hopf as well as an operadic perspective to the theory of hyperplane arrangements and reflection arrangements (and more generally to the theory of finite semigroups). Along with giving new proofs of known results, it has led to several new concepts such as the category of lunes, Lie and Zie elements, and noncommutative zeta and Möbius functions which we first presented in [**10**]. We plan to explain further ideas such as operadic homology for arrangements in future work.

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