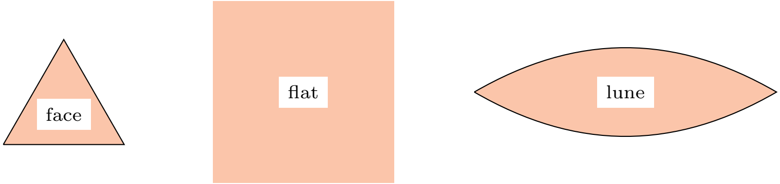
Introduction

We now describe the contents of the monograph in more detail, along with pointers to important results. For organizational purposes, the book has been divided into three parts; they are preceded by an introductory chapter on reflection arrangements.

Reflection arrangements. (Chapter 1.) A linear hyperplane arrangement $\mathcal{A}$ is a finite collection of hyperplanes in a real vector space passing through the origin. It has many associated geometric objects such as faces, chambers, flats, bifaces, lunes, bilunes, and so on (Table 1.1). Some illustrative pictures are shown below.



Unlike faces and lunes, flats do not have a boundary; in fact they are subspaces, so the middle picture is to be interpreted as extending indefinitely to the plane of the paper.

The set of faces, flats, bifaces carry monoid structures called the Tits monoid, Birkhoff monoid, Janus monoid, respectively. Linearizing them over a field $\Bbbk$ yields the Tits algebra, Birkhoff algebra, Janus algebra, respectively. We also construct certain incidence algebras called the flat-incidence algebra, lune-incidence algebra, bilune-incidence algebra. They contain various interesting kinds of zeta and Möbius functions.

We say $\mathcal{A}$ is a *linear reflection arrangement* if it is preserved by reflection in any of its hyperplanes (assuming an inner product on the ambient vector space). The group generated by these reflections is finite and is called the *Coxeter group* of $\mathcal{A}$. We denote it by W. By taking W-orbits of faces, flats, bifaces, lunes, and so on, we obtain notions of face-types, flat-types, biface-types, lune-types, and so on (Table 1.2). We introduce the groupoid of biface-types whose objects are face-types and morphisms are biface-types (Section 1.7). A wealth of enumeration identities related to face-types and shuffles are given in Sections 1.8 and 1.9. Similarly, we have the W-invariant subalgebras of all algebras mentioned above. We call them the invariant Tits algebra, invariant lune-incidence algebra, and so on (Sections 1.10 and 1.12).

The braid arrangement, which is an important example of a reflection arrangement, is reviewed in Section 1.16.

Parts I and II

We give a combined introduction to Coxeter species and Coxeter spaces. They are formulated in terms of faces and face-types, respectively; Table III summarizes the main notations. Hopf theories of these two objects proceed largely in parallel with each other. We bring out these similarities, often by putting concepts next to each other in tabular form. At the same time and more importantly, we also point out subtle differences between the two.

Table III. Coxeter species and Coxeter spaces.

Faces			Face-types		
Coxeter species	p, q	β, w	Coxeter spaces	V, W	σ
Coxeter monoids	a, b	μ_A^F	Coxeter algebras	A, B	μ_Z^T
Coxeter comonoids	c, d	Δ_A^F	Coxeter coalgebras	C, D	Δ_Z^T
Coxeter bimonoids	h, k	μ_A^F, Δ_A^F	Coxeter bialgebras	H, K	μ_Z^T, Δ_Z^T

Coxeter species and Coxeter spaces. (Chapters 2 and 8.) Fix a reflection arrangement $\mathcal{A}$ with Coxeter group W.

Coxeter species. A *Coxeter species* p is a family of vector spaces $\mathsf{p}[F]$, one for each face F, together with linear maps

$$\beta_{G,F} : \mathsf{p}[F] \to \mathsf{p}[G] \quad \text{and} \quad w : \mathsf{p}[F] \to \mathsf{p}[wF],$$

called support and type morphisms, the former whenever F and G have the same support, and the latter for each F and $w \in W$, subject to suitable compatibility axioms, see (2.12a) and (2.12b). A *Coxeter monoid* a is a Coxeter species together with linear maps

$$\mu_A^F : \mathsf{a}[F] \to \mathsf{a}[A],$$

one for each $A \leq F$, subject to naturality, associativity, unitality axioms, see (2.28a) and (2.28b). A *Coxeter comonoid* c is defined dually using linear maps

$$\Delta_A^F : \mathsf{c}[A] \to \mathsf{a}[F]$$

for $A \leq F$, subject to naturality, coassociativity, counitality axioms, see (2.36a) and (2.36b). We refer to μ as the product and to Δ as the coproduct. A *Coxeter bimonoid* h is a Coxeter species which carries a Coxeter monoid and a Coxeter comonoid structure, and satisfies the bimonoid axiom, see (2.39). It involves the Tits product on faces. The data in a Coxeter bimonoid is shown in Summary 2.1.

The above notions can also be formulated in terms of flats and lunes, see Summary 2.2. Further, they have commutative analogues obtained using the (co)commutativity axiom, see (2.49) and (2.57) for faces and (2.52) and (2.58) for flats and lunes. We also define a *Coxeter q-bimonoid* for a scalar

q by replacing the bimonoid axiom by the q-bimonoid axiom, see (2.42) and (2.45). The q-distance function on faces plays a starring role here. For $q = -1$, we use the term signed Coxeter bimonoid.

Coxeter spaces. A *Coxeter space* V is a family of vector spaces $\mathrm{V}[T]$, one for each face-type T, together with linear maps

$$\sigma : \mathrm{V}[T] \to \mathrm{V}[U],$$

one for each biface-type σ, subject to suitable compatibility axioms (involving composition of biface-types), see (8.28). A *Coxeter algebra* is a Coxeter space together with linear maps

$$\mu_Z^T : \mathrm{V}[T] \to \mathrm{V}[Z],$$

one for each $Z \leq T$, subject to naturality, associativity, unitality axioms, see (8.41). There is a dual notion of a *Coxeter coalgebra*, and a self-dual notion of a *Coxeter bialgebra*. The bialgebra axiom involves a sum and is not set-theoretic, see (8.64). It involves biprojection of biface-types which again has the Tits product at its core. The data in a Coxeter bialgebra is shown in Summary 8.3.

The above notions can also be formulated in terms of top-nested faces starting with support and type morphisms

$$\beta_{G,D,F,C} : \mathrm{V}[F,C] \to \mathrm{V}[G,D] \quad \text{and} \quad w : \mathrm{V}[F,C] \to \mathrm{V}[wF, wC],$$

and so on, see Summary 8.1. Another possibility is to formulate them in terms of top-lunes, see Summary 8.2. Further, they have commutative analogues obtained using the (co)commutativity axiom, see (8.79) and (8.88) for top-nested faces, (8.82), (8.83) and (8.90) for top-lunes, (8.86) and (8.91) for face-types. We also define a *Coxeter q-bialgebra* for a scalar q by replacing the bialgebra axiom by the q-bialgebra axiom, see (8.66), (8.69), (8.73). For $q = -1$, we use the term signed Coxeter bialgebra.

Coxeter species vs Coxeter spaces. Coxeter species carry more structure than Coxeter spaces. To appreciate this point, suppose $u, v \in W$ are such that $u(F) = v(F) = G$ and $u(C) = D$ and $v(C) = E$ as shown in Figure II.

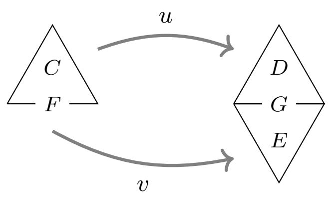


Figure II. Coxeter species vs Coxeter spaces.

For a Coxeter species p, we have $u, v : \mathsf{p}[F] \to \mathsf{p}[G]$, while for a Coxeter space V, we have $u : \mathrm{V}[F,C] \to \mathrm{V}[G,D]$ and $v : \mathrm{V}[F,C] \to \mathrm{V}[G,E]$. The key observation is: In the first case, u and v link the same two objects, while in the second case, they link different objects.

Duality functor. Duality was mentioned in the above discussion. More formally: There is a duality functor on the category of Coxeter species which interchanges Coxeter monoids and Coxeter comonoids, and preserves Coxeter q-bimonoids (Sections 2.3.8 and 2.5.9). There is a similar duality functor on the category of Coxeter spaces which interchanges Coxeter algebras and Coxeter coalgebras, and preserves Coxeter q-bialgebras (Sections 8.4.10 and 8.7.8).

Signature functors. There are ways to connect the unsigned and signed worlds. For Coxeter species, we have the u-signature functor and o-signature functor, see Section 2.8. They are related to signature spaces under and over flats, respectively. For Coxeter spaces, we have the u-signature functor, see Section 8.10. The additional structure present in a Coxeter species allows for two signature functors as opposed to only one for Coxeter spaces.

Functor categories. A salient feature of the categories of Coxeter monoids, Coxeter comonoids, Coxeter bimonoids is that they are functor categories just like the category of Coxeter species (Section 2.9). The same holds for the categories of Coxeter (co, bi)algebras (Section 8.11). This is possible due to the absence of the traditional tensor product of vector spaces in the construction of these categories.

Bimonads and bimonad bialgebras. The preceding notions can be put into a categorical framework using monads, comonads, bimonads (Appendix A.1). There exists a bimonad $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$ on the category of Coxeter species whose bialgebras are Coxeter bimonoids. Similarly, there exists a bimonad $(\overline{\mathcal{T}}, \overline{\mathcal{T}}^\vee, \overline{\lambda})$ on the category of Coxeter spaces whose bialgebras are Coxeter bialgebras. These results along with their companions are stated in Propositions 2.71 and 8.115. For commutative aspects, see Propositions 2.78 and 8.121. We mention in particular the bimonads $(\mathcal{S}, \mathcal{S}^\vee, \lambda)$ and $(\overline{\mathcal{S}}, \overline{\mathcal{S}}^\vee, \overline{\lambda})$.

Examples. (Chapters 4 and 10.) We now turn to examples.

Coxeter bimonoids. The *exponential Coxeter bimonoid* E is defined by setting $\mathsf{E}[A] := \Bbbk$ for all faces A, and all structure maps

$$\beta_{G,F} : \mathsf{E}[F] \to \mathsf{E}[G], \qquad w : \mathsf{E}[F] \to \mathsf{E}[wF],$$
$$\mu_A^F : \mathsf{E}[F] \to \mathsf{E}[A], \qquad \Delta_A^F : \mathsf{E}[A] \to \mathsf{E}[F]$$

to be identities. The bimonoid axiom holds trivially. More generally, for a W-module M, we have the *decorated exponential Coxeter bimonoid* E_M whose components are all M; structure maps involving β, μ, Δ are identities, while those involving w employ the action of w on M. See Tables 4.1 and 4.2 for closely related objects.

More examples of Coxeter bimonoids based on chambers, faces, flats, top-nested faces, top-lunes, pairs of chambers are given in Chapter 4.

Coxeter bialgebras. The *Coxeter bialgebra of polynomials* $\Bbbk[x]$ is defined by setting $\Bbbk[x][T] := \Bbbk$ for all face-types T. The linear maps σ are all identities. For clarity, we write x^T for the basis element $1 \in \Bbbk[x][T]$. The product and coproduct are

$$\begin{aligned} \mu_Z^T : \Bbbk[x][T] &\to \Bbbk[x][Z] & \qquad \Delta_Z^T : \Bbbk[x][Z] &\to \Bbbk[x][T] \\ x^T &\mapsto x^Z & x^Z &\mapsto d_{T/Z}\, x^T, \end{aligned}$$

where $d_{T/Z}$ is the number of faces of type T greater than some fixed face of type Z. (For the braid arrangement, $d_{T/Z}$ is a product of multinomial coefficients.) More generally, we have the Coxeter q-bialgebra of polynomials $\Bbbk_q[x]$ obtained by replacing $d_{T/Z}$ by a suitable face-type enumeration polynomial $d_{T/Z}(q)$ (which generalizes the Poincaré polynomial of W). The verification of the q-bialgebra axiom for $\Bbbk_q[x]$ is nontrivial and equivalent to the gate identity (1.59). As companions to $\Bbbk[x]$, we define the *divided power Coxeter bialgebra* $\Bbbk\{x\}$ and *signed Coxeter bialgebra of dual numbers* $\Bbbk[dx]$, see Table 10.1.

More generally, for a W-module M, we define the *tensor Coxeter q-bialgebra* $\overline{\mathcal{T}}_q(M)$, *shuffle Coxeter q-bialgebra* $\overline{\mathcal{T}}_q^{\vee}(M)$, and *symmetric Coxeter bialgebras* $\overline{\mathcal{S}}(M)$ and $\overline{\mathcal{S}}^{\vee}(M)$, *exterior signed Coxeter bialgebras* $\overline{\mathcal{E}}(M)$ and $\overline{\mathcal{E}}^{\vee}(M)$, see Table 10.3. The W-action on M by shuffles enters into (co)product formulas. The q-bialgebra axioms for $\overline{\mathcal{T}}_q(M)$ and $\overline{\mathcal{T}}_q^{\vee}(M)$ are equivalent to identities (1.69) in the group algebra of W.

More examples of Coxeter bialgebras based on face-types, flat-types, Coxeter symmetries are given in Chapter 10.

Face-type enumeration. (Sections 1.8 and 1.9.) As noted above, (co)products of Coxeter bialgebras are quite intricate and tied to ideas from face-type enumeration. A systematic summary of such connections is given in Table IV. Identities in the left column are used to prove results in the right column. Interestingly, the logical flow can be reversed. That is, one can first prove results in the right column via the technology of universal constructions or Fock functors, and then use them to deduce identities in the left column.

TABLE IV. Coxeter bialgebras and face-type enumeration.

Face-type enumeration	Relevance to Coxeter bialgebras
polynomials (1.38), (1.41), (1.52)	Observation 8.99 (Section 8.9)
multiplicative property (1.32)	Section 10.2.4, Lemma 10.12, Section 10.2.17
gate identity (1.59)	Lemma 9.89, Section 10.2.4
gate identity (1.57), identity (1.40)	Section 10.4.4
gate identity (1.58)	Lemma 10.79
alternating sign identities (1.63)	Lemma 9.130, antipode formulas (10.30), (10.116), (10.117)
multiplicative property (1.66), identities (1.69), (1.70), (1.71)	Section 10.3.6
alternating sign identity (1.72)	antipode formulas (10.57) and (10.58)

The above phenomena are not seen for Coxeter bimonoids where the bimonoid axiom is set-theoretic and easier to handle. This simplicity happens because of the extra structure present in faces in comparison to face-types which keeps enumeration polynomials hidden from view.

Shuffles. (Section 1.6.) For any face-type T, we have the notion of T-shuffles. These are Coxeter symmetries obtained by taking Tits product of a reference chamber C_r with faces of type T.

Shuffles become visible in the study of Coxeter bialgebras when we focus on the S-component, where S is the maximum face-type. The starting point is that $\mathrm{V}[S]$ is a W-module for any Coxeter space V (Exercise 8.38). The action of T-shuffles on $\mathrm{H}[S]$ is present in the q-bialgebra axiom of H. We also already noted that shuffles appear in (co)product formulas of the tensor and shuffle Coxeter q-bialgebras; this is because they are "freely generated" by their S-components. See Table V for precise pointers.

Table V. Coxeter bialgebras and shuffles.

T-shuffles, T-deshuffles	(1.21), (1.22), (1.23)
(co)projection of biface-types	Exercise 1.35
q-bialgebra axiom	Exercises 8.73 and 8.76
tensor and shuffle Coxeter q-bialgebras	formulas (10.37) and (10.39)

Hopf theory for reflection arrangements. (Chapters 3 and 9.) We develop the basic theory of Coxeter bimonoids and Coxeter bialgebras discussing primitive and decomposable filtrations, universal constructions, abelianization and coabelianization, Hadamard product and enrichment, exp-log correspondences, norm transformation, characteristic operations, antipode. Pointers to important formulas and results are summarized in Table VI. This can be compared with the theory for bimonoids for hyperplane arrangements developed in [**11**, Part II]. Some specific points are highlighted below.

Species vs Coxeter species. The bimonoid and (co)commutativity axioms for a Coxeter species do not involve type morphisms. The same is true for maps involved in exp-log correspondences. As a consequence, results on Coxeter species can be formally deduced from corresponding results on species proved in [**11**], see Remark 2.52. In contrast, for Coxeter spaces, both support and type morphisms intervene in the bialgebra and (co)commutativity axioms. This explains why in Table I in the preface, in the last column, there is no analogue of species and bimonoids (involving only support morphisms).

The connection between species and Coxeter species can be made precise using semidirect products, see (2.67) and Lemma 2.70. This is analogous to the connection between vector spaces and W-modules. There is more that one can do with W-modules than with vector spaces. The same is true for Coxeter species. For instance, type morphisms in a Coxeter species allow us to connect it to Coxeter spaces via Fock functors.

TABLE VI. Hopf theory for Coxeter species and Coxeter spaces.

Hopf theory	Coxeter species	Coxeter spaces
Cauchy powers	Formula (3.1)	Formula (9.1)
com. Cauchy powers	Formula (3.4)	Formulas (9.4), (9.5)
primitive and decomposable filtrations	Table 3.1	Table 9.1
	(3.9) and (3.12)	(9.9) and (9.12)
primitive and indecomposable part functors	Lemma 3.5	Lemma 9.8
universal constructions and adjunctions	Table 3.2	Table 9.2
	Summary 3.1	Summary 9.1
(co)abelianization adjunctions	Summary 3.2	Summary 9.2
Hadamard product	Table 3.3	Table 9.3
	Formula (3.53)	Formula (9.59)
enrichment via internal hom	Section 3.6.3	Section 9.6.3
enrichment via universal measuring	Section 3.7.4	Section 9.7.4
exp-log correspondences ((co)derivation version)	Table 3.4	Table 9.4
	Theorem 3.37	Theorem 9.56
	Theorem 3.43	Theorem 9.61
exp-log correspondences (series version)	Theorem 3.56	Theorem 9.76
	Theorem 3.59	Theorem 9.80
	Lemma 3.66	Lemma 9.85
pq map	Proposition 3.44	Proposition 9.65
logarithm of identity	Proposition 3.46	Proposition 9.67
	Proposition 3.50	Proposition 9.72
general q-norm map	Theorem 3.79	Theorem 9.110
characteristic operation	Formula (3.149)	Formula (9.155)
two-sided characteristic operation	Formula (3.157)	Formula (9.164)
antipode	Formula (3.159)	Formula (9.171)
rank one (toy example)	Section 7.8	Section 13.8

Universal constructions. There is a complete parallel between universal constructions for Coxeter species and Coxeter spaces. These constructions follow from the existence of the bimonad $(\mathcal{T}, \mathcal{T}^\vee, \lambda)$ on Coxeter species and bimonad $(\overline{\mathcal{T}}, \overline{\mathcal{T}}^\vee, \overline{\lambda})$ on Coxeter spaces. Thus, $\mathcal{T}(\mathsf{p})$ is the free Coxeter monoid on the Coxeter species p, while $\overline{\mathcal{T}}(\mathrm{V})$ is the free Coxeter algebra on the Coxeter space V. Explicitly,

$$\mathcal{T}(\mathsf{p})[A] = \bigoplus_{F:\, A \leq F} \mathsf{p}[F] \quad \text{and} \quad \overline{\mathcal{T}}(\mathrm{V})[Z] = \bigoplus_{T:\, Z \leq T} \mathrm{V}[T],$$

with product components μ_A^F and μ_Z^T given by inclusion. Cofree objects are constructed dually using $\mathcal{T}^\vee$ and $\overline{\mathcal{T}}^\vee$. Going one step further, these free and cofree objects are used to build universal Coxeter bimonoids and Coxeter bialgebras by using the mixed distributive laws λ and $\overline{\lambda}$.

For commutative aspects, we need to bring in the monads $\mathcal{S}$, $\overline{\mathcal{S}}$ and comonads $\mathcal{S}^\vee$, $\overline{\mathcal{S}}^\vee$. Thus, $\mathcal{S}(\mathsf{p})$ is the free commutative Coxeter monoid on the Coxeter species p, and $\overline{\mathcal{S}}(\mathrm{V})$ is the free commutative Coxeter algebra on the Coxeter space V, and so on.

Hadamard product. Coxeter bimonoids possess certain features not to be seen for Coxeter bialgebras. The key distinction is that the bimonoid axiom is set-theoretic, while the bialgebra axiom is not. As a result, Hadamard product preserves Coxeter bimonoids, but it does not preserve Coxeter bialgebras in general. This yields the *decoration functor* on Coxeter species obtained by taking Hadamard product with E_M (Section 4.9) which has no analogue for Coxeter spaces. Similarly, we have the biconvolution Coxeter bimonoid built out of two Coxeter bimonoids, but no analogous construction for Coxeter bialgebras. See Remarks 9.35 and 9.38. See also Theorem 7.12 which is specific to Coxeter bimonoids.

Exp-log correspondences. Interestingly, exp-log correspondence (say between primitive and group-like series) is not unique in general; it depends on the choice of an invariant noncommutative zeta and Möbius function. The same remark applies to logarithm of the identity map. There are two situations where we do have uniqueness, namely, the bicommutative setting and the q-setting for q not a root of unity. In these situations, the logarithm of the identity map induces an isomorphism from the indecomposable part to the primitive part, and its inverse is the pq map. The latter is always defined (that is, with no dependence on any noncommutative zeta or Möbius function).

There is no difference between the free Coxeter algebra and free commutative Coxeter algebra on one generator. In contrast, the free Coxeter monoid and free commutative Coxeter monoid on one generator differ. In our notations,

$$\mathcal{T}(\mathsf{x}) = \Gamma \neq \mathsf{E} = \mathcal{S}(\mathsf{x}) \quad \text{and} \quad \overline{\mathcal{T}}(\mathrm{X}) = \Bbbk[x] = \overline{\mathcal{S}}(\mathrm{X}).$$

This gives rise to two notions of exponential series for Coxeter monoids, but only one notion for Coxeter algebras.

Characteristic operations. Coxeter bimonoids admit characteristic operations by faces, while Coxeter bialgebras admit characteristic operations by face-types. To say this precisely, recall that the Tits algebra has a basis indexed by faces, while the invariant Tits algebra has a basis indexed by face-types. For a cocommutative Coxeter bimonoid h and cocommutative Coxeter bialgebra H,

$$F \cdot x := \mu_O^F \Delta_O^F(x) \quad \text{and} \quad T \cdot x := \mu_\emptyset^T \Delta_\emptyset^T(x),$$

respectively, define a left action of the Tits algebra on $\mathsf{h}[O]$ and of the invariant Tits algebra on $\mathrm{H}[\emptyset]$. Here O is the smallest face, and $\emptyset$ is the smallest face-type.

Moreover, for h as above, the Tits algebra action on $\mathsf{h}[O]$ combined with the W-action on $\mathsf{h}[O]$ via type morphisms yields an action of the Coxeter–Tits algebra on $\mathsf{h}[O]$. (The Coxeter–Tits algebra is the semidirect product of the action of the Coxeter group on the Tits algebra.) In fact, this leads to an equivalence between categories of cocommutative Coxeter bimonoids and left Coxeter–Tits algebra modules (Proposition 3.86). This and other similar results are summarized in Proposition 3.82.

q not a root of unity. We prove many results for Coxeter q-bimonoids and Coxeter q-bialgebras under the assumption that q is not a root of unity. This assumption while convenient to state is stronger than what is required. Since our theory is local to a reflection arrangement, the results remain valid provided we stay away from certain specific roots of unity, namely, the zeroes of the determinants of the Varchenko matrices in Theorem 1.56 and its proof (which ensures that the two-sided q-zeta and q-Möbius functions exist). For instance, in rank one, $q \neq \pm 1$ suffices, see [**11**, Remark 13.78] in this regard.

Coxeter operads. (Chapters 5 and 11.) A Coxeter dispecies $\mathbf{p}$ can be formulated using flats or top-lunes or face-types as a family of vector spaces

- $\mathbf{p}[\mathrm{X}, \mathrm{Y}]$, one for each pair of flats $\mathrm{X} \leq \mathrm{Y}$, or
- $\mathbf{p}[\mathrm{L}, \mathrm{M}]$, one for each pair of top-lunes $\mathrm{L} \leq \mathrm{M}$, or
- $\mathbf{p}[T, U]$, one for each pair of face-types $T \leq U$.

In each of these three setups, the components are connected by linear maps subject to suitable compatibilities. This richness of setups is specific to the Coxeter context and not to be seen for dispecies considered in [**11**, Chapter 4].

The category of Coxeter dispecies is a monoidal category under the substitution product, see formulas (11.10), (11.11), (11.15). Monoids wrt this product are Coxeter operads. The substitution map of a Coxeter operad is summarized in Table 11.1. (It can be compared to multiplication of upper triangular matrices.)

Coxeter operad monoids and Coxeter operad algebras. The category of Coxeter species is a left module category over the category of Coxeter dispecies via (5.9). (This structure can be compared to how an upper triangular matrix acts on a column vector.) Thus, for any Coxeter operad $\mathbf{a}$, we have the category of left $\mathbf{a}$-modules in Coxeter species. We call it the category of Coxeter $\mathbf{a}$-monoids.

Similarly, the category of Coxeter spaces is a left module category over the category of Coxeter dispecies via any of the equivalent formulas (11.25), (11.28), (11.29). Thus, for any Coxeter operad **a**, we have the category of left **a**-modules in Coxeter spaces. We call it the category of Coxeter **a**-algebras. The structure map of a Coxeter **a**-algebra is summarized in Table 11.2.

Coxeter commutative, associative, Lie operads. The three fundamental examples of Coxeter operads are the Coxeter commutative, associative, Lie operads which we denote **Com**, **As**, **Lie**, respectively. We develop them in all three setups: flats, top-lunes, face-types. Each setup has its advantage and appeal. A Coxeter **As**-monoid is the same as a Coxeter monoid, while a Coxeter **As**-algebra is the same as a Coxeter algebra. Such results along with connections to monad algebras are summarized in Tables 5.1 and 11.4.

In contrast to **Com** and **As**, it is considerably harder to formulate **Lie** and establish its presentation as a binary quadratic structure. The core work for this was done in [**10**, Sections 10.6 and 14.5], [**11**, Example 4.12].

Lie theory for reflection arrangements. (Chapters 6 and 12.) Along with Coxeter (co, bi)monoids, we also have the notion of a Coxeter Lie monoid. Similarly, we have the notion of a Coxeter Lie algebra. These are Coxeter **Lie**-monoids and Coxeter **Lie**-algebras, respectively, associated to the Coxeter operad **Lie**.

Table VII. Lie theory for Coxeter species and Coxeter spaces.

Lie theory	Coxeter species	Coxeter spaces
(co)commutator (co)bracket	Formula (6.4)	Formula (12.10)
	Formula (6.7)	Formula (12.15)
primitive and indecomposable parts	Proposition 6.1	Proposition 12.3
	Proposition 6.2	Proposition 12.4
(co)free Lie construction	Theorem 6.3	Theorem 12.7
	Theorem 6.5	Theorem 12.12
universal (co)enveloping construction	Theorem 6.8	Theorem 12.21
	Theorem 6.11	Theorem 12.26

Pointers to important concepts and results are summarized in Table VII. This can be compared with the theory for Lie monoids for hyperplane arrangements developed in [**11**, Chapter 16].

Binary operations vs higher operations. The product maps μ_A^F and μ_Z^T in a Coxeter monoid and Coxeter algebra are "higher operations" arising from elements of the Coxeter associative operad **As**. More traditionally, using the presentation of **As**, one can consider "binary operations", that is, μ_A^F and μ_Z^T only when $A \lessdot F$ and $Z \lessdot T$ subject to the covering associativity axiom. The same can be done for **Com** and **Lie**. (The Lie case is much harder